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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

ne **Association of Mathematics Teachers of India (AMTI)** was started 965 for the promotion of efforts to improve Mathematics education at all ;ls. A major aim of the Association is to assist practicing teachers of ıthematics in schools in improving their expertise and professional skills. nother important aim is to spot out and foster Mathematical talent in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centers in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds "Inter State Mathematical Talent Search Competition" annually and organises Orientation Courses, Seminars and Workshops for teachers and summer courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education has been instituted by Prof. R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official journal of the Association and is published quarterly. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in three languages – English, Hindi and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for the MT and effective from April 1993:

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The Mathematics Teacher

(INDIA)

OFFICIAL JOURNAL OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Volume: 38 Issues: 3 & 4 Year: 2002

Prof. A.Ramakrishnan

Editor

TORIAL

In this part of volume 38 of Mathematics Teacher containing the issues 3 and 4, the papers presented at the One-day seminar on Calculus conducted on 10-11-2001 at I.I.T. (Madras), Chennai sponsored by AMTI and Abacus Software Ltd., Chennai are being published. Two papers presented at the thirty sixth Annual Conference of AMTI held in December, 2001 at Cochin also appear in this issue. Professor S.Sathik's paper continues the discussion on Jarasandha numbers and suggests some interesting points for further research. Prof. Litwiller and Prof. Duncan have discussed in their paper how Binomial Statistical Situation helps in analysing Preference Polling.

We are sorry to report the sad demise of Prof. J.N.Kapur, President of AMTI from 1977 to 1989 at Delhi on 4-9-2002. An article on him appears in this issue.

Our current President Prof. R.C.Gupta has been elected as an effective member of the International Academy of the History of Science, Paris. We offer our hearty congratualations to him and wish him many more honours.

We aslo express our thanks to the Department of Science and Technology, Ministry of Science and Technology, Government of India for a grant of Rs.30,000/- towards meeting the publication cost of Mathematics Teacher partially.

To function effectively money is necessary. May we solicit the philanthropic minded well wisher of AMTI to generate funds through donations to a larger forum of lovers of Mathematics.

Donations to The Association of Mathematics Teachers of India **(AMTI)** are exempt from Income Tax u/s **80G** of the Income Tax Act 1961 by letter No.

DIT (E) No. 212 (285)/73 dated **31.10.2002**

Secretary, AMTI.

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Contributions are invited for publication in *Mathematics Teacher*. The writers are requested to send their articles written legibly (or typed) on one side of the paper only with figures, if any, drawn neatly in separate sheets. If acknowledgement of the receipt of papers is needed, a self-addressed post card is to be enclosed. It will not be possible to return papers which are not accepted for publication.

Editor, *Mathematics Teacher*.

SOME NEW STUDIES AND FINDINGS REGARDING ANCIENT INDIAN MATHEMATICS [1]

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A comprehensive and authentic History of Mathematics in India from antiquity to modern time is yet to be written. Attention of scholars to this national need was drawn during an invited talk I delivered in the 53^{rd} annual conference of the Indian Mathematics Society. The lecture "*Swami Vidyaranya* and the History and Historiography of Mathematics in India" was at once published in the *Mathematics Student* (1987). But nothing seems to have been done in this connection so far.Actually the task will involve some sort of a team work and the motivation for the initiation or proposing has to be welcomed and looked favourably by organizations such as IMS,UGC,NBHM (under AEC), Commission on History of Science (under INSA), etc. Whether such a project will become fruitful soon or not, is a separate matter. Till then the keen scholars and students of readers - category will have to keep satisfied with available older and present books on the subject. To supplement that material I propose to give a brief and rapid survey of some new studies and findings in the field.

[1] Presidential address at the 36th Annual Conference of AMTI held at Cochin in December 2001.

In spite of sincere efforts and great claims, the Indus Valley script has not been deciphered successfully. And so, only conjectures continue at the best, and in some matters, controversies further increase. For example, the Indus script symbol U has been equated to numeral 100 (J.E.Mitchiner), 20 (B.V.Subbarayappa), 5 (S.C.Kak), and to the Tamil Letter *WU* (B.Z.Szalek)! In a paper published (1984) in the Interim Report of the Field Work done by the *Aachen University*, E.Maula interpreted the great ring stones as calendaric instruments. A remarkable Indus seal (old No. M430) depicts one central and seven minor 'dvinities' which have been variously interpreted as 7 *Sapta-rishis*, 7 *old planets*, 7 *krittikas* and even 7 *chief amatya of Risabhadeva*, the first *Jaina Tirthankar*! It seems that these variants reflect only biased views and we are far away from the actual situation.

The well-known Pythagoras theorem, named after Greek philospher - mathematician of the 6^{th} century B.C. is now believed to be known earlier not only in India but also in Babylonia, China etc. A.Seidenberg has shown that the Vedic Taittiriya Samhita knew not only the algebraic or computational aspect of the theorem but also its geometric aspect. But in respect of Vedas and Vedic Corpus, there seems to be no moderation in regard to highly fantastic claims of advanced scientific achievements. These include the proof of the famous Fermat's Last Theorem and the Einsteinian mass-energy equation $E = mc^2$

As the popular verse puts -

शिक्षा कल्पोऽथ व्याकरणं
निरुक्तं वृन्दसां चयः ।
ज्योतिषामयनं चैव
वेदाङ्गानि षडेव तु ॥

Siksha kalpo atha vyakaranam
niruktham chandasam chayaha
jyothishamayanam chaiva
vedangani shadeva thu

there are six *Vedangas*, namely *Siksa* (phonetics), *Kalpa* (ritualistics), *Vyakarana* (grammar), *Nirukta* (etymology), *Chandas* (Prosody), and *Jyotisa* (astronomy and astrology). Mathematical aspects of some of them have been studied afresh. M.D.Pandit's papers and book on Zero in Panini have illustrated that the concept of mathematical zero had philosophical and linguistic background. The techniques used by Panini (about 500 B.C.) are comparable to those adopted in mathematics, computers, symbolic logic and artificial intelligence.

Sulba-sutras come under the Kalpa and form the oldest geometrical handbooks of India. By now the names of a dozen such books are known. Baudhayana Sulba Sutra, the oldest of them, is available in print with two commentaries (Varanasi 1979). A good edition of Apastamba Sulba Sutra with 4 commentaries has been brought out by Dr. Damodar Jha whose edition of the Maitrayania Sulba Sutra is also just out (Hoshiarpur 2001).

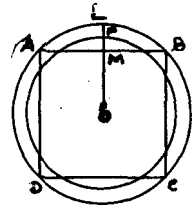

Fig. 1 - Vedic circling of Square

The famous Vedic problem of circling a square demands a handy measure of the radius OP such that the smaller dotted circle (Fig.1) may be practically equal to the given square $ABCD$ in area. The usual Sulba method is to take

$$OP = OM + ME/3$$

The present speaker interpreted a line in the Manava Sulba Sutra to yield (Gupta 1988)

$$OP = OE - \dot{O}E/5$$

which leads to $\pi = 25/8$ which is best approximation so far found in the Sulbas.

A noteworthy recent study of ancient Vedic mathematics is the bold hypothesis of the ritual origin of mathematics put forward by A. Seidenberg who supported the diffusion theory for significant mathematical results. Making a sort of synthesis of Seidenberg's ideas, B.L. Van der Waerden has proposed a central Eurasian origin of mathematics before the dispersion of Aryans.

Ancient Indian numeration was decinary. Vedic works contain a few lists of decuple terms including that of Medhatithi upto the parardha $(= 10^{12})$ and its extension

upto loka (= 10^{19}). Biggest such list was designed by Buddhists. It consists of 60 terms ending with asamkhyaloka (= 10^{59}). But 8 intermediary terms were soon lost. Recently I reconstructed the full list with the help of suitable terms from other works. It is relevant to mention that the longest list of decuple terms found in the world is that of 97 terms in Amalasiddhi, a late work.

Among the non-decimal scales, the Valmiki Ramayana's laksa-scale has been completed upto Mahaugha = 10^{60} by consulting fuller text than in Srinivasiengar's book. The centesimal scale of Lalitavistara upto tallaksana = 10^{53}, is well-known from 23 terms beyond koti (crore). According to K.Menninger, the eight more technical terms found in the work represent eight more such series. Thus the full counting-scale will go upo

$$10^{7+9\times46} = 10^{421}$$

a monstrous number indeed! However, his conjecture lacks literary support.

The Jaina School of Indian mathematics has many peculiarities. Their cosmography needed mensuration of a circle and its segment. Their very long calculation of the circumference and area of the Jambu Island upto a micro unit (where one yojana = 5.3×10^{16} such units) has been now successfully explained (Gupta, 1975). A number of ancient Indian rules for the rectification of a circular segment (like Bharata-varsa) of chord c and height h have been shown to follow the proto-type (Gupta, 1979)

$$\text{arc} = \sqrt{c^2 + kh^2}, \text{ where } k = \pi^2 - 4$$

Research on Aryabhata the elder (born A.D. 476) has been going on ever since Kern edited the Aryabhatiya (Leiden, 1874). We are lucky that several editions and translations are now available. The cliometrical analysis of the work by Roger Billard demonstrates that Aryabhata made his own independent observations (i.e. his astronomical constants and parameters are not borrowed from the Greeks). The fractional parts of his tabular sine-differences ($R = 3438, h = 225$ minutes) have been found (Gupta, 1971) in a supercommentary by Govindasvamin (about A.D.800-850). By giving proper interpretations to certain Sanskrit technical terms, it has been now shown that Aryabhata I knew the correct method for finding the altitude of the Sun at any time of the day (Gupta, 1977). But his nativity of Kerala has been now challenged by K.V. Sarma.

Varahimihira (6^{th} century A.D.) is a great name in Indian jyotisa-sastra and his Brhatsamhita is historically important. It is shown by Takao Hayashi that in connection with 'combinations of perfumes' (a chapter in the above work), Varahamihira had utilized a pandiagonal magic square of order four, probably the one which is depicted in Fig. 2.

10	3	13	8
5	16	2	11
4	9	7	14
15	6	12	1

Fig. 2.

The same work also contains in a condensed form a rule (purvena purvena gatena yuktam etc.) which enables us

to compute the binomial coefficients by a tabular numerical method (called lostha prastara). But he missed the Pascal triangle here.

Bhaskara I was a great follower of Aryabhata I and not a direct disciple as was thought earlier. His commentary (A.D.629) on the Aryabhatiya has brought to light several things about ancient Indian mathematicians Maskari, Purana and Mudgala whose works are now lost. Prof. K.S.Shukla's pioneering work on Bhaskara I (for which he got the D.Litt. degree) has been published in the editions of the latter's Mahabhaskariya, Laghu-Bhaskariya and commentary on Aryabhata's work. The first two also include English translation of the text. Translation of the Ganita-pada of the commentary has been made recently by Agathe T. Keller as her doctoral thesis (Paris VII Univ., 2000). Bhaskara I's remarkable approximation

$$\sin A = \frac{4A(180 - A)}{40500 - A(180 - A)} \ , A \text{ in degree}$$

has been shown to be very popular in India.

Two mathematical chapters from Brahma-sphuta Siddhant (A.D. 628) of Brahmagupta were translated by Cobbrooke as early as in 1816 but full English translation of the work is still a challenge. Brahmagupta's rules about cyclic quadrilaterals, including its area and diagonals, are praise-worthy. So also is his contribution to the solution of the equation

$$Nx^2 + I = y^2 \quad \text{(called Varga} - \text{prakrti)}$$

Recent work by C.O.Selenius and V.Murthy on the famous cakravala, or Indian cyclic method, and its modifications

further glorify India's marvellous achievements in the field long before such things were attained in Europe.

The Bakhshali Mathematics is now believed to be of 7^{th} century. The work is never found quoted or mentioned in other works and may be just an exercise - book meant for training as the colophon indicates. The ironical fact is that the manuscript of this work on Indian mathematics was found in a place now in Pakistan, deposited in U.K. studied by a Japanese for doctoral work in U.S.A., and finally, now, published from Holland in an excellent form (after long gap from Kaye's work). Channabasappa found that the Bakhshali Square-root formula is more efficient than Newton's algorithm.

Study of ancient works on prosody or metrical science has brought to light some results of mathematical interest. For example in connection with matra-vrttas, the *Vrttajati-samuccaya* of Virhanka (between 6^{th} and 8^{th} centuries) gives the rule

$$U_{n+2} = U_n + U_{n+1}$$

which defines the Fibonacci number.

It seems that the 8th century in India began with quite a rich heritage of traditional mathematics. Sridhara did the consolidation and wrote suitable text-books on the subject. A recent discovery of Sridhara's rule in a work of Govinda Svamin confirms that the former may be safely placed in the middle of the 8^{th} century. Unfortunately his Patiganita is not extant in full-form but has been edited and translated by Shukla (whose dating on the work is, however, wrong).

The consolidation and exposition of the Mathematics of the Jaina School was done by Virasena, Mahavira (9^{th} century) and continued by Nemicandra (10^{th} century). A new edition of Mahavira's Ganita sarasangraha with M.Rangachari's 1912 translation and a Kannada translation is published (Hombuja, 2000). Mahavira's work is a "collection" and preserver of some old material including the newly discovered Babylonian rule ($s = c + h$) for arc of a circular segment. Better interpretations are now available for rule for surface of a spherical segment and for his excellent generalization of Brahmagupta's rule for volumes of frustum-like solids.

Jaina concept of Vargita - Samvargita defined as the self powerial function

$x\rfloor^n$, where $x\rfloor = x\rfloor^1$, $x\rfloor^2 = (x\rfloor)^{x\rfloor} = (x^x)^{(x^x)}$; etc. was needed in their definition of infinities. With its help, a new lower bound for the first unenumerable number has been obtained (Gupta 1992).[2]

I have mentioned only a few new findings about ancient Indian Mathematics. The question of the dividing line between ancient and medieval periods is there. One view is that the establishment of Delhi Sultanate caused decadence of indigenous creative art and culture and marked the end of ancient period. The southern India was less affected and we indeed have the achievements of the Late Aryabhata School. They were neglected half a century ago but now history of Mathematics (whether written in India or abroad) usually has noteworthy place for them. S.Parameswaran's recent

[2] $x\rfloor^1 = x^x = y$ say. Then $x\rfloor^2 = y^y = z$ say. Then $x\rfloor^3 = z^z$, and

book the Golden Age of Indian Mathematics (1998) rightly glorifies the achievements.

The most important publication in our area is the comprehensive multi-volume Census of the Exact Sciences in Sankrit (Philadelphia 1970-1994) and the best doctoral thesis of the last decades is Y.Ohashi's A History of Astronomical Instruments in India (1992). Latest thesis is that on Sridhara (Rani Durgavati Univ., 2001). Among the areas still needing attention are kundas, yantras, and prosodic works.

We believe that history of any subject represents the human and humanity aspects of the subject, and should be its parts. The History of Mathematics should form part of Mathematics. Education in Mathematics must include a knowledge of the development of Mathematics. Study of History of Mathematics helps in understanding Mathematics and also enriches teaching of Mathematics.

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The AMTI

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in our society.

THE ENORMITY OF ZERO [1]

Prof. George Gheverghese Joseph

University of Manchester
United Kingdom University of Exeter,
United Kingdom University of Toronto, Canada.

A. Introduction

A few years ago on a British television programme I was asked: "Why did Zero originate in India?" Fortunately, I was allowed enough time to develop an answer without assuming as most television programmes do today that the audience watching have the attention span of a grasshopper. Trying to gather my thoughts, I resorted to the familiar ploy of taking refuge in definitions. If zero merely signified a magnitude or a direction separator (i.e. separating those above the zero level from those below the zero level), the Egyptian zero, nfr, dating back at least four thousand years, amply served these purposes. If zero was merely a place-holder symbol, indicating the absence of a magnitude at a specified place position (such as, for example, the zero in 101 indicates the absence of any "tens" in one hundred and one), then such a zero was already present in the Babylonian number system long before the first recorded occurrence of the Indian zero. If zero was represented by just an empty space within a well-defined positional number system, such a zero was present in Chinese mathematics a few centuries before the Indian zero. The absence of a symbol for zero did

[1] Text of R.C.Gupta History of Mathematics Endowment lecture given at the 36th Annual Conference of AMTI held at Cochin in December 2001.

not prevent it from being properly integrated into an efficient computational tool that could even handle solution of higher degree order equations involving fractions. However, the Indian zero alluded to in the question was a multi-faceted mathematical object: a symbol, a number, a magnitude, a direction separator and a place-holder, all in one operating within a fully established positional number system. Such a zero occurred only twice in history -the Indian zero which is now the universal zero and the Mayan zero which occurred in solitary isolation in Central America around the beginning of Common Era.

To understand the first appearances of the Indian and Mayan zeroes, it is necessary to examine them both within the social contexts in which the two independent inventions occurred. At the same time we should attempt to identify certain common threads in both cultures that led to the occurrence of the zero in only these two cultures. The dissemination of the Indian zero as a part and parcel of the Indian numerals is one of the more remarkable episodes in the History of Mathematics. But what is rarely recognised is that this transmission occurred through a number of cultural and linguistic filters which may have inhibited a clearer understanding of the concept of zero and the arithmetic of the operations with zero. Because of the popular difficulties with the zero, there has occurred over time a series of avoidance mechanisms to cope with the presence of zero, which have far-reaching pedagogical implications. And these include the general absence of any discussion at the educational level of the topic of 'calculating with zero' ("shunya ganita"), which was emphasized in practically all-Indian texts on Mathematics from the time of Brahmagupta (b. AD 598) onwards. This is a serious deficiency in the

Mathematics curriculum both in schools and colleges and needs urgent rectification. As illustrations of this deficiency, consider how the following questions will be answered by students of Mathematics:

Is zero a positive or negative number?

Is zero an odd or even number?

Divide 2 by zero

I have found that even among the university students of Mathematics, a discussion of these three questions tend to be confused. And it is my experience that an approach through History provides an effective and interesting way of introducing this difficult subject. Incidentally, History is also effective in introducing Non-Euclidean Geometry at the university level.

B. The History of Zero: The Indian Dimension

The word 'zero' comes from the Arabic "al-sifr". Sifr in turn is a transliteration of the Sanskrit word "shunya" meaning void or empty, which became later the term for zero. It was introduced into Europe during Italian Renaissance in the 12th century by Leonardo Fibonacci (and by Nemorarius a less well-known mathematician) as "cifra" from which emerged the present 'cipher'. In French, it became "chiffre", and in German "ziffer", both of which mean zero.

The ancient Egyptians never used a zero symbol in writing their numerals. Instead they had a zero to represent a value or magnitude. A bookkeeper's record from the 13th Dynasty (about 1700 BC) shows a monthly balance

sheet for items received and disbursed by the royal court during its travels. On subtracting total disbursements from total income, a zero remainder was left in several columns. This zero remainder was represented by the hieroglyph, nfr, which also means beautiful, or complete in ancient Egyptian. The same nfr symbol also labeled a zero reference point for a system of integers used on construction guidelines at Egyptian tombs and pyramids. These massive stone structures required deep foundations and careful leveling of the courses of stone. A vertical number line labeled the horizontal leveling lines that guided construction at different levels. One of these horizontal lines, often at pavement level, was used as a reference and was labeled nfr or zero. Horizontal leveling lines were spaced 1 cubit apart. Those above the zero level were labeled as 1 cubit above nfr, 2 cubits above nfr and so on. Those below the zero level were labeled 1 cubit, 2 cubits, 3 cubits, and so forth, below nfr. Here zero was used as a reference for directed or signed numbers.

It is quite extraordinary that the Mesopotamian culture, more or less contemporaneous to the Egyptian culture which had developed a full positional value number system on base 60 did not use zero as a number. A symbol for zero as a place-holder appeared late in the Mesopotamian culture. The early Greeks, who were the intellectual inheritors of Egyptian Mathematics and Science emphasised Geometry to the exclusion of everything else. They did not seem interested in perfecting their number notation system. They simply had no use for zero. In any case, they were not greatly interested in Arithmetic, claiming that Arithmetic should only be taught in democracies for it dealt with relations of equality". On the other hand, Geometry was the natural

study for oligarchies for "it demonstrated the proportions within inequality."

In India, the zero as a concept probably predated zero as a number by hundreds of years. The Sanskrit word for zero, shunya, meant "void or empty". The word is probably derived from shuna, which is the past participle of svi, "to grow". In one of the early Vedas, Rgveda, occurs another meaning: "the sense of lack or deficiency". It is possible that the two different words, were fused to give "shunya" a single sense of "absence or emptiness" with the potential for growth. Hence, its derivative, Shunyata, described the Buddhist doctrine of Emptiness, being the spiritual practice of emptying the mind of all impressions. This was a course of action prescribed in a wide range of creative endeavours. For example, the practice of Shunyata is recommended in writing poetry, composing a piece of music, in producing a painting or any activity that comes out of the mind of the artist. An architect was advised in the traditional manuals of Architecture (the Silpas) that designing a building involved the organisation of empty space, for "it is not the walls which make a building but the empty spaces created by the walls." The whole process of creation is vividly described in the following verse from a Tantric Buddhist text:

> First the realisation of the void (shunya),
> Second the seed in which all is concentrated
> Third the physical manifestation
> Fourth one should implant the syllable

The mathematical correspondence was soon established. "Just as emptiness of space is a necessary condition for the

appearance of any object, the number zero being no number at all is the condition for the existence of all numbers".

A discussion of the mathematics of the shunya involves three related issues: (i) the concept of the shunya within a place-value system, (ii) the symbols used for shunya, and (iii) the mathematical operations with the shunya. Material from appropriate early texts are used as illustrations below.

It was soon recognised that the shunya denoted notational place (place holder) as well as the "void" or absence of numerical value in a particular notational place. Consequently all numerical quantities, however great they may be, could be represented with just ten symbols. A twelfth century text (Manasollasa) states:

"Basically, there are only nine digits, starting from 'one' and going 'nine'. By the adding of zeros these are raised successively to tens, hundreds and beyond."

And in a commentary on Patanjali's Yogasutra (7th century A.D.) there appears the following analogy:

"Just as the same sign is called a hundred in the "hundreds" place, ten in the "tens" place and one in the "units" place, so is one and the same woman referred to (differently) as mother, daughter or sister."

The earliest mention of a symbol for zero occurs in the Chandahsutra of Pingala (fl. 3rd century BC) which discusses a method for calculating the number of arrangements of long and short syllable in a metre containing a certain number of syllables (ie., the number of combinations of two items from a total of n items, being

allowed). The symbol for shunya began as a dot (bindu), found in inscriptions both in India and in Cambodia and Sumatra around the seventh and eighth century and then became a circle (chidra or randra meaning a hole). The association between the concept of zero and its symbol was already well-established by the early centuries of the Christian era, as the following quotation shows: "The stars shone forth, like zero dots (shunya-bindu) – scattered in the sky as if on the blue rug, the Creator reckoned the total with a bit of the moon for chalk" (Vasavadatta, ca A.D. 400).

A Sanskrit text on Mathematics / Astronomy from the time of Brahmagupta usually contains a section called "shunya-ganita" or computations involving zero. While the discussion in the arithmetical texts (patiganita) is limited only to the addition, subtraction and multiplication with zero, the treatment in algebra texts (bijaganita) covered such questions as the effect of zero on the positive and negative signs, division with zero and more particularly the relation between zero and infinity (ananta).

Take as an example, Brahmagupta's seventh century text Brahmasphuta-Siddhanta. In it he treats the zero as a separate entity from the positive (dhana) and negative (rhna) quantities, implying that shunya is neither positive nor negative but forms the boundary line between the two kinds, being the sum of two equal but opposite quantities. He stated that a number, whether positive or negative, remained unchanged when zero is added to or subtracted from it. In multiplication with zero, the product is zero. A zero divided by zero or by some number become zero. Likewise the square and square root of zero is zero. But

when a number is divided by zero, the answer is an undefined quantity "that which has that zero as the denominator."

The earliest inscription in India of a recognizable antecedent of our numeral system is found in an inscription from Gwalior dated 'Samvat 933' (AD 876). The spread of these numerals westwards is a fascinating story. The Arabs were the leading actors in this drama. Indian numerals probably arrived at Baghdad in 773 AD with the diplomatic mission from Sind to the court of Caliph al-Mansur. In about 820 A.D. al-Khwarizmi wrote his famous Arithmetic, the first Arab text to deal with the new numerals. The text contains a detailed exposition of both the representation of numbers and operations using Indian numerals. Al-Khwarizmi was at pains to point out the usefulness of a place-value system-incorporating zero, particularly for writing large numbers. Texts on Indian reckoning continued to be written and by the end of the eleventh century, this method of representation and computation was widespread from the borders of Central Asia to the southern reaches of the Islamic world in North Africa and Egypt.

In the transmission of Indian numerals to Europe, as with almost all knowledge from the Islamic world, Spain and (to a lesser extent) Sicily played the role of intermediaries, being the areas in Europe, which had been under Muslim rule for many years. Documents from Spain and coins from Sicily show the spread and the slow evolution of the numerals, with a landmark for its spread being its appearance in an influential mathematical text of medieval Europe, Liber Abaci, written by Fibonacci (1170-1250) who learnt to work with Indian numerals during his extensive travels in

North Africa, Egypt, Syria and Sicily. And the spread westwards continued slowly, displacing Roman numerals, and eventually, once the contest between the abacists (those in favour of the use of abacus or some mechanical device for calculation) and the algorists (those who favoured the use of the new numerals) had been won by the latter, it was only a matter of time before the final triumph of the new numerals occurred with bankers, traders and merchants adopting the system for their daily calculations.

C. The History of Zero: The Mayan Dimension

Evidence relating to Pre-Columbian Maya civilisation comes from three main sources: four screen-fold books called codices, a large number of stone monuments and thousands of ceramic vessels. The best account of the Maya culture around the time of the Spanish Conquest comes from a Franciscan priest, Diego de Landa who recorded the history and traditions of the Maya people around 1566. Piecing together these different strands of evidence, it is possible to construct an account of the social context in which the Mayan numerals and especially the Mayan zero emerged around the beginning of the Christian era.

The Mayan system of numerical notation was one of the most economical systems ever devised. In the form that was used mainly by the priests for calendar computation as early as 400 BC, it required only three symbols. A dot was used for one, and a bar for five; and a symbol for zero which resembles a snail's shell. With these three symbols they were able to represent any number on a base 20. However, there was an unusual irregularity in the operation of the place value system. Corresponding to

our units, tens, hundreds, thousands, ... etc, the Mayans had units, 20's, (18×20)'s, (18×202)'s, (18×203)'s, .. etc. This anomaly reduces the efficiency in arithmetical calculation. For example, one of the most useful facilities in our number system is the ability to multiply a given number by 10 by adding a zero to the end of it. An addition of a Mayan zero to the end of a number would not in general multiply the number by twenty because of the mixed base system employed. This inconsistency also inhibited the development of further arithmetical operations, particularly those involving fractions.

To understand this curious irregularity in Mayan numeration, it is important to appreciate the social context in which the number system was used. As far as we know this form of writing numbers was used only by a tiny elite - a group of priest scribes who were responsible for carrying out astronomical calculations and constructing calendars.

At the top of the pyramid was a hereditary leader who was both a high-priest (Ahau-Can) and a Maya noble. Under him were the master scribes who were priests as well as teachers and writers ("engaged in teaching their sciences as well as in writing books about them"). Mathematics was recognised as such an important discipline that depictions of scribes who were adept at that discipline appear in the iconography of Mayan artists. Their mathematical identity was signified in the manner in which they were depicted: either with the Maya bar and dot numerals coming out of their mouths or a number scroll being carried under their armpit. The location of the scroll under the armpit with numbers written on it would seem a status

symbol. In an interesting illustration on another Maya vase of from the beginning of the Christian era, there is a seated supernatural figure with the ears and hooves of a deer, attended by a number of human figures, including a kneeling scribe mathematician from whose armpit emanates a scroll containing the sequence of numbers 13, 1, 3, 3, 4, 5, 6, 7, 8 and 9. At the top right hand corner of this illustration there is the small figure of a scribe who looks female, with a number scroll under her armpit indicating that she is a mathematician and possibly the one who painted the scene and wrote the text on the vase. She is described as Ah T'sib ("the scribe"). Preceding this text is a glyph that has not been deciphered but which could be her name. Once the name is deciphered, and if the scribe is female, we may have the name of one of the earliest known women mathematician-scribe in the world. The existence of female mathematicians / scribes among the Mayans is further supported by another depiction found on another ceramic vase. The text on this vessel contains the statement of the parentage of the scribe in question: "Lady Scribe Sky, Lady Jaguar Lord, the Scribe". Not only does she carry the scribal title at the end of her name phrase but she incorporates it into one of her proper names, an indication of the importance she herself places on that reality.

Returning to the curious irregularity in the Mayan place value system, the general view is that it is tied to the exigencies of operating three different calendars. The first calendar, known as the tzolokin or 'sacred calendar', contained 260 days in twenty cycles of 13 days each. Superimposed on each of the cycles was an unchanging series of twenty days, each of which was considered a god to whom

prayers and supplications were to be made. The second, known as a civil or secular calendar, was the one for practical use. It was a solar calendar consisting of 360 days grouped into 18 monthly periods of twenty days and an extra month consisting of five days. The last month was shown by a hieroglyph that represented disorder, chaos and corruption and any one born in that month was supposed to have been cursed for life. Finally, there was the third calendar of 'long counts' similar to the Indian 'Yuga' periodisation. The upper section of one of the oldest standing stelas at Ires Zapotes in Mexico shows the date of its construction in the calendar of 'long counts' as:

8 kins	=	8×1	=	8 days
		(20 kins	=	1 uinal)
16 uinals	=	20×16	=	320 days
		(18 uinal	=	1 tun)
0 tuns	=	$20 \times 18 \times 0$	=	0 days
		(20 tuns	=	1 katun)
6 katuns	=	$(18)202 \times 6$	=	7206 days
		(20 katuns	=	1 baktun)
16 baktuns	=	$(18)203 \times 16$	=	2304000 days
		(20 baktuns	=	1 piktun)
7 piktuns	=	$(18)204 \times 7$	=	20160000 days
		(20 piktuns	=	1 calabtun)

TOTAL 22,471,534 days which corresponds to 31 BC.

There were higher units of measurement, notably

kinchiltuns (or kins) and alautins where 1 alautin equalled 23,040,000,000 days.

Measurement of time constituted a central feature of the Mayan culture and the interest in measurement was carried into Mayan Astronomy. We can only marvel at the high degree of accuracy that the Mayans achieved in their astronomical work. To illustrate, without any sophisticated equipment and with the deficiency of a mixed base system, they obtained the mean duration of a solar year as 365.242 days (modern value: 365.242198 days) and the mean duration of a lunar month as equivalent to 29.5302 (modern value: 29.53059 days)

D. The Two Zeroes: Common Threads and Differences

I began this talk with the question relating to the Indian zero, which has now been extended to include the Mayan Zero. Why did the full use of zero within a well-established positional value system only emerge in two cultures? Were there any similarities between the two cultures that might provide an answer, however tentative it remains?

From the existing evidence, much of it fairly fragmentary especially in the Mayan case, we are aware that both cultures were numerate with considerable interest in Astronomy. The Indian culture from an early time showed interest and even fascination for large numbers and there is no contrary evidence to indicate that this was not so in the Mayan culture. Both cultures were obsessed with the passage of time but in different ways. The Indian interest was tied up with the wide-spread belief in a never-ending cycle of

births and rebirths with the primary objective for individual salvation being the need to break the cycle. This was apparently achieved during the Vedic times by carrying out sacrifices on specially constructed altars, which conformed to specific shapes and sizes and where the sacrifices had to be carried out on particular days chosen for their astronomical significance. In the Mayan case, the obsession took the form of a society's fear that the world would come to an end unless the gods (and especially the Sun God) were propitiated by human sacrifice to be undertaken systematically at certain propitious time of the year to be dictated by specific astronomical occurrences. In both cases there was need for accurate measurement of time and hence the detailed calendars and the elaborate periodisation into eras. The need for such precise calculations may have stimulated the development of efficient number systems with a fully developed zero. And it was probably only an accident of History and Geography that the Indian zero prevailed while the Mayan zero eventually disappeared into oblivion.

E. The Consequences

As mentioned earlier, the spread of the Indian zero had to go through a number of cultural and linguistic filtering processes, the imperfect nature of which is evident even in popular culture today. Culturally, our discomfort with the concepts of zero (and infinite) is often reflected in humour. Underlying such uneasiness is both a conceptual fuzziness regarding zero and a lack of confidence in the manipulation of mathematical expressions where the notions of zero or infinity present themselves. A story told of youthful Srinivas Ramanujan illustrates this point well. In an Elementary

Mathematics class the teacher was explaining the concept of division (or 'sharing') through examples. If three bananas were shared between three children, each child would get one banana. And similarly, the share would be one banana if four bananas were divided among four children, five bananas among five children and so on. And when the teacher generalised this idea of sharing out x bananas among x boys, Ramanujan piped up with a question: If x equalled zero, would each child then get one banana? There is no record of the teacher's reply.

Consider another illustration of the widespread conceptual ambiguity relating to zero. There appeared the following item on a German television news program in 1977: Smog alarm in Paris: Only cars with an odd terminating number on the license plate were allowed on the roads in Paris. Cars with an even such number were not allowed to be driven. There was a problem: Is the terminating number 0 an even number? Drivers with such numbers were not fined, because the police did not know the answer!

Ask a mathematician whether zero is an even or an odd number. The answer would be: If you define evenness or oddness on the integers (either positive or all), then zero should be taken to be even; but if you define evenness and oddness on the natural numbers, then zero would be neither. This is because we apply concepts such as "even" only to "natural numbers," in connection with primes and factoring, where by "natural numbers" one means positive integers and so excludes zero. However, those who work in the area of the foundations of Mathematics consider zero a natural number, and for them the integers are whole numbers. From that

point of view, the question whether zero is even just does not arise, except by extension. One may say that zero is neither even nor odd. Because you can pick an even number and divide it in groups, take, e.g., 2, which can be divided in two groups of "1", and 4 can be divided in two groups of "2". But can you divide zero? This was basically Ramanujan's question. And the difficulty is basically caused by the fact that the concepts of even and oddness predated zero and the negative integers.

On the question of division by zero there seems to be confusion found even among seasoned practitioners. In a recent widely-used American textbook on Operations Research, I came across the division of 3 by zero in a Simplex tableau involving the familiar 'column ratio test', with the conclusion, $3 \div 0 =$ infinity (??). However, the author refused to follow the logic of this conclusion by continuing the Simplex calculation based on this result. Neither did the book detect the erroneous nature of its reasoning by asking the obvious question: Which number, when multiplied by zero, gives you 3? Infinity?! But then infinity is not a number; it is a concept. Another commonly given answer is: 'Undefined'. Is this correct? Not really! To find the correct answer, look for it from the 'mouth of babes' or even from a pocket calculator: One cannot ever meaningfully divide by zero. For if one allow division by zero, then one enters a topsy-turvy world where $1 = 2$! Zero is a number but it is not similar to other numbers when the arithmetical operation of division is involved.

Given these difficulties, one response to zero is to avoid the term itself and use 'euphemisms' such as 'nought', 'O',

'nothing', etc. In reciting a telephone number, a postal zip code or a street number or any of a variety of other numerical codes, we try to avoid the use of the name "zero". All the other digits are correctly enunciated. In tennis scores, zero is called "love," because zero looks like an egg. The French called it "l'oeuf," which was corrupted to 'love'. Zero is placed as the last number on a computer keyboard after all other digits. It often appears at the bottom of the keypad on a telephone. There is even a resistance to zero in labelling the ground level of a building as the '0' level. However, it is interesting where such a resistance is absent, as for example in the case of certain buildings in Eastern Europe (or in South America), the practice is to label floors as $-1, 0, 1, 2, 3, \cdots$, with -1 representing the basement. This practice is instructive. It signifies that in the absence of a concept of zero there could have been only positive numerals. The incorporation of zero in Mathematics opened up the new dimension of negative numerals. Incidentally, it is precisely because negative numbers seem to have first appeared in Chinese Mathematics that the distinguished Chinese historian of Mathematics, Lam Lay Yong, has argued that the zero was invented by the Chinese - a symbol-less zero in this case.

It is as though the name 'zero' itself invokes a kind of psychological anxiety perhaps associated with "nothingness", a kind of emptiness which humankind finds uncomfortable and prefer to avoid confronting. As with all such anxiety-provoking ideas, some other imagery is substituted which provides a veneer to mask the disquieting emotional undertones of the discomforting idea. Yet in the two cultures where zero was first used as a number within

a well-developed positional value system, the concept was likely devoid of any negative overtones, just as the word 'nefr' had positive connotations in ancient Egypt.

Finally, there are certain semantic issues relating to zero that often remain unexamined. The concept of zero has been associated with terms such as 'nothing' or 'emptiness' or 'void'. But an interesting question arises: Is the presence of nothing (reflecting non-existence) different from the absence of something or anything (reflecting non-availability)? "Not there" reflects that the number or item(s) exists but they are just not available. "Nothing" however reflects non-existence. To muddy the semantic pool further, there are whole shades of meaning associated with the term 'zero' depending on whether it is used as a noun, a verb, an adverb, and even an adjective as in "zero possibility". For example, using the Americanism, "We zeroed in on the cause," means we had identified all the possibilities, and have discovered the one that was pertinent to our investigation. In this use as a verb, zero may be said to equal one. However, the statement, "the result was a big, fat, zero" signifies "nothing". Here, zero has the quality of not being there, providing an illustration of how conceptual ambiguity in ordinary speech tends to make the comprehension of the mathematical meaning even more difficult.

As teachers of Mathematics at all levels, we should be aware of the varied nature of the difficulties faced by students at various levels of their mathematical education confronted by zero. Should we follow the example of the early Indian mathematics and bring 'shunyaganita' back into the Mathematics curriculum?

ON A PAPER BY P.S.N.SASTRY[1]

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The author deserves appreciation for having brought out some interesting properties of Jarasandha numbers. The writer wishes to add some points that would throw some more light. The author formulated the J number as

$$XC \rightarrow (X+C)^2 = XC \tag{1}$$

$$XC = (X+C)^2 = 10^n X + C \tag{2}$$

The equation (2) is the general form of the J numbers. The simple numerical illustration of the author is

$$\begin{aligned} 81 &\rightarrow (8+1)^2 = 9^2 = 81, \; n = 1 \\ 9801 &\rightarrow (98+01)^2 = 99^2 = 9801, \; n = 2 \\ 998001 &\rightarrow (998+001)^2 = 999^2 = 98001, \; n = 3 \end{aligned}$$

Equation (2) is quadratic with two unknowns X and C. However any value of C cannot yield the desired X for a J number. The author has stated a condition - without systematically deriving - that for $n =_{1,2,3}$ etc the numbers shown below in square brackets [ ] should be a perfect square.

$$[25 - 9C], [2500 - 99C], [250000 - 99C], \cdots \tag{3}$$

[1] Jarasandha Numbers published in The Mathematics Teacher, Vol.37 (3 and 4) pp.151-162, 2001.

The basis for this mere statement is not clear in the first reading, since it is implicit. In order to make it explicit, the following modified form of equation (2) is derived by adding and subtracting $[10^n C]$ and rewriting the quadratic for $(X + C)$.

$$(X + C)^2 - 10^n(X + C) + (10^n - 1)C = 0 \qquad (4)$$

The solution of this is well known.

$$(X + C)_{1,2} = 10^n/2 \pm [(10^n/2)^2 - (10^n - 1)C]^{\frac{1}{2}} \qquad (5)$$

If the roots are to be integers the second term on the right side must be an integer. In other words the quantity in the square bracket [ ] must be a perfect square. Let this integer number be A. Therefore

$$A^2 = (10^n/2)^2 - (10^n - 1)C \qquad (6)$$

so that the roots are

$$(X + C)_{1,2} = (10^n/2 + A) \text{ and } (10^n/2 - A) \qquad (7)$$

Now inserting $n =_{1,2,3}$ in equation (6) we get the numbers stated by the author as in expression (2). Equation(6) could be rewritten also in two other forms as

$$(10^n/2)^2 - A^2 = (10^n - 1)C \qquad (8)$$

and

$$(10^n/2 + A)(10^n/2 - A) = (10^n - 1)C \qquad (9)$$

It can be seen that the factors on the left side of equation (9) are the two roots and the right side is an expression for the product of the roots. This also follows from the properties of polynomial / quadratic equations - see equation (4) - the

sum of the roots is the coefficient of the variable $(X + C)$ i.e., 10^n and the product is the constant i.e., $(10^n - 1)C$. Two of the three findings of the author simply follow from these properties. Elaborate numerical illustration is not required when this fact is recognized. The third finding that one of these two roots is invariably a multiple of 11 is not conspicuous from all the expressions. However the writer notices one interesting point. In equations (4) and (9), the product of the roots which are $9C, 99C, 999C, 9999C$ etc should be definitely a multiple of 9. This in turn suggests that one of the roots could be a multiple of 9. The products of roots for $n = 2, 4, 6, \cdots$ (even) which are $99, 9999, 999999$, etc are definitely multiples of 11. For odd numbers of n, it is not so. From this point of view the author's numerical illustrations are very valuable, since they bring out these hidden properties.

The writer notices another interesting point related to the roles of 9 and 11. When one of the roots is a multiple of 9, the same root or the second root is a multiple of 11. When the same root is a multiple of both 9 and 11, the other root is a multiple of neither 9 nor 11. This can be verified from the author's numerical pairs of roots: First case: $(55, 45), (2728, 7272), (2223, 7777)$. Second case: $(279, 703), (4950, 5050)$.

The solution of the equation (6) is not straight forward as the author has also stated, since there are two unknowns with one equation, that too with a stringent condition (6). However certain bounds for C could be studied in order to limit the trial values for C. A value of 1 for C is obvious as a lower bound with A and (roots) as 4 and (1,9); 49 and (1,99),499 and (1,999) etc. For

even numbers of n, the upper bounds are with A, (roots) and C as $5,(55,45)$ and $25;50,(5050,4950)$ and $2500;500,(500500,499500)$ and 250000, etc. The general expression for C thus is $10^n/4$. A cursory glance of C for odd numbers of n also indicates that this upper bound could be valid. For example for $n = 3, A = 203$, roots $= 297$ and $703, C = 209.C$ is below 250. We can therefore choose the bounds for C as

$$1 \leq C \leq 10^n/4 \tag{10}$$

There are no other conditions visible that narrow down the bounds or ranges of bounds. Wizards in number theory should be able to throw more light on this.

For the associated J number problem /

$$XC \rightarrow (X - C)^2 = 10^n X - C = XC \tag{11}$$

the modified quadratic equation is

$$(X - C)^2 - 10^n X - (10^n + 1)C = 0 \tag{12}$$

Its solution is

$$(X - C)_{1,2} = 10^n/2 \pm [(10^n/2)^2 + (10^n + 1)C]^{\frac{1}{2}} \tag{13}$$

The second term on the right side in the square brackets is greater than $10^n/2$. Hence one of the roots is always negative. Here also $C = 1$ is the lower bound. But the upper bound cannot be established because of the positive sign for $(10^n+1)C$ in equation (13), which is the product of the roots. For all integral values of C, the products of roots should be multiple of $(10^n + 1)$ i.e., $11;101;1001;10001$ etc. For $n = 1,3,5,\cdots$ the products are multiples of 11. Hence one of the two roots is likely to be a multiple of 11.

The author's finding that the positive roots are multiples of 11, therefore, is consistent with the above since he has considered only the case of $n = 3$.

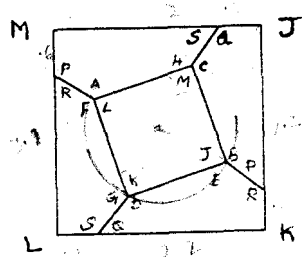
It is not clear why the author has computed the product of the square numbers while the product of the numbers itself is adequate for the study and is indicative of the property that the product must be a negative number $= -(10^n + 1)C$ as can be seen in equation (12). It appears to a mechanical engineer like me that a lot more work is possible in this area for serious researchers in number theory. Perhaps the author himself could take up this challenging task.

Corrigendum

In \$1 of the paper "On Some cardboard Constructions" by Shri H.S. Keshavamoorht (Mathematics Tecaher, Vol. 37, Issues 3 & 4, 2001, p169) kindly read 'octagon' instead of 'hexagon' (both in the title of the article and in the first line of the text). With reference to the second paragraph of \$1, the following figure may be found helpful. The error is regretted.

(Editor, MT.)

Figure



PREFERENCE POLLING: A BINOMIAL STATISTICAL SITUATION

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Teachers are constantly looking for situations in which probability ideas can be exemplified. We will discuss such a situation involving a binomial distribution.

Suppose that 60% of a university faculty favours a certain curricular proposal and that 40% are opposed. A newspaper wishes to take a "straw-poll" by surveying a random sample of five faculty members before the official vote. What is the probability that this straw-poll will correctly predict the actual outcome vote of the entire faculty?

Under what circumstances would the newspaper poll lead to the correct prediction of passage in the actual faculty vote? This outcome would be correctly predicted if 3,4, or 5 of the polled faculty members favored the proposal. What is the probability that this polling result occurs?

To solve this problem, the binomial random variable concepts must be employed, where:

- n = the number of independent trials, each trial resulting either in a success or a failure.
- p = the probability of success for each trial.

- q = the probability of failure for each trial $(p+q = 1)$.
- X = the number of successes occurring in the n trials.

The probability of achieving exactly "r" successes among these n independent trials is then $C(n,r)p^r q^{n-r} = \frac{n!}{r!(n-r)!}p^r q^{n-r}$.

To adapt this general binomial model to our polling situation, regard the selection of each polled faculty member as an independent trial. For each such selection, let "success" be the outcome in which the polled faculty member favors the proposal, and "failure" be the outcome in which he / she is opposed to the proposal. The number of persons in the sample who favor the proposal is denoted by X.

For the specific situation being considered, $n = 5; p = 0.6; q = 0.4$. Then,

$$\begin{aligned} P(X = 3, 4, \text{or } 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= C(5,3)(0.6)^3(0.4)^2 + C(5,4)(0.6)^4(0.4)^1 \\ &\quad + C(5,5)(0,6)^5(0.4)^0 \\ &= \frac{5!}{3!(2!)}(0.6)^3(0.4)^2 + \frac{5!}{4!(1!)}(0.6)^4(0.4)^1 \\ &\quad + \frac{5!}{5!(0!)}(0.6)^5(0.4)^0 \\ &= 0.3456 + 0.2592 + 0.07776 \\ &= 0.68256 \end{aligned}$$

This is a fairly high probability of correctly predicting the

outcome of the faculty vote; but, is only slightly better than the 60% likelihood that would result if only one voter was polled. Note that the probability of the poll's incorrectly predicting the election result is $1 - 0.68256 = 0.31744$; this can also be calculated as $P(X = 0, 1, \text{ or } 2)$.

The editor of the newspaper reasons that a more reliable prediction might result if a larger sample were used. Consequently, it is decided to poll a random sample of six faculty members, rather than five. In this new situation, what is the probability that the poll will correctly predict the election outcome?

The outcome will be correctly predicted if 4,5, or 6 of the polled voters support the proposal. The probability of this occurring is: $P(X = 4) + P(X = 5) + P(X = 6)$

$$\begin{aligned} &= \frac{6!}{4!(2!)}(0.6)^4(0.4)^2 + \frac{6!}{5!(1!)}(0.6)^5(0.4)^1 \\ &\quad + \frac{6!}{6!(0!)}(0.6)^6(0.4)^0 \\ &= 0.31104 + 0.186624 + 0.046656 \\ &= 0.54432 \end{aligned}$$

why is this probability less than the approximately 68.3% resulting from the use of a random sample of size five? The reason is that the use of six polled voters introduces the possibility of a tie among those polled. When this occurs, the newspaper cannot make a prediction, since they have no evidence that either side is more likely to prevail than the other. The probability of this tie is $P(X = 3) = \frac{6!}{3!(3!)}(0.6)^3(0.4)^3 = 0.27648$.

Let us summarize the results of the "$n = 6$" poll:

- P(newspaper correctly predicts the election result) $= P(X = 4, 5,$ or $6) = 0.54432$.
- P(newspaper is unable to make a prediction) $= P(X = 3) = 0.27648$.
- P(newspaper incorrectly predicts outcome) $= P(X = 0, 1, or 2) = 1 - 0.54432 - 0.27648 = 0.1792$.

As a final example, suppose that the newspaper decides to poll a random sample of 11 faculty (choosing an odd number to avoid a tie possibility). In this situation ($n = 11$), the probability that the newspaper correctly predicts the election outcome is:

$$P(X = 6, 7, 8, 9, 10, \text{ or } 11) = 0.220724 + 0.236490$$

$$+0.177367 + 0.088684 + 0.026605 + 0.003628 = 0.753498.$$

The probability of an incorrect prediction is then

$$P(X = 0, 1, 2, 3, 4, or 5) = 1 - 0.753498 = 0.246502.$$

Challenges:

1. How many faculty should be included in the random sample so that the probability of a correct prediction exceeds 90%; 95%; 99% ?
2. For a new value of p, re-examine this polling problem for different values of n.
3. Find other circumstances in which a binomial probabilistic analysis is appropriate.

SOME THOUGHTS ON TEACHING ANALYTICAL GEOMETRY AND CALCULUS [1]

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$ 1. Introduction

Mathematics Education ought to (a) improve skills (b) foster critical thinking and, above all, to (c) encourage non routine problem solving. If students are encouraged to attend to these we, as teachers, will have done our duty.

$ 2. Skills

(a) Essentially, the skills deal with (i) the art of curve sketching, (ii) (relative) *max/min* and rate problems (iii) area and volumes. Each of us has his/her own favourite(s) in these. Let us talk about only a few of these in some detail.

Curve Sketching without Calculus

Usually this is done for polynomials f by locating the x-intercept, y-intercept and by checking if the graph of $y = f(x)$ crosses or touches the x-axis. Some students have a rough time understanding it. It seems to me a result of too much logical (" If p implies q and q implies r then... etc") as opposed to geometric training / thinking on their part resulting in a weak geometric intuition of such and other

[1] Lecture delivered at the One day seminar on Calculus held on 10-11-2001 at I.I.T. (Madras), Chennai.

things. You might object, "Wait a minute; does not Calculus make this easier?". Of course you are right and we all know the power of Calculus. But it is better for students to be thrown on their ingenuity. For example (without Calculus) suppose a is a zero of a certain polynomial; how do we tell if the graph touches the x-axis at a and crosses it at b? Let us also look at $x = c$ where it seems that f has a (relative) maximum; how can we find (at least an approximate value of) the maximum $f(c)$? Cannot we use a calculator to locate c by repeated bisection?

With calculus

Of course we have definite and systematic rules of procedure. Once again it is essential to get students to understand the ideas behind intervals where f increases / decreases, of concavity, points of inflection and of (relative) max/min.

Intuitive rather than rigorous.

At a first introduction to these, it is better to do them suggestively rather than rigorously. After all, most of the applications we deal with are polynomials, rational functions with a mild dose of Trigonometry, exponential and (natural) logarithm functions. To be honest we should mention that there are situations, even in applications, where you need to proceed with caution.

Maxima / Minima

This is a place where Calculus excels and its power can and must be shown to students via applications. We might

if time (that inscrutable creature in John L Kelley's words) permits show some examples where it is not easy to find the critical points algebraically. For example, think about the positive x in the equation, $f'(x) = \sin x - 0.5x = 0$. Now where is this "$x$"? Is it less than $\pi/2$? More than it and less than π? Why? Next, can we find x approximately? If so, how? (we need not prove any general theorem in Analysis or Numerical Methods).

Areas and Volumes

This topic, as we all know, needs not just skills of integration but also some geometric understanding however elementary it may seem to us. One of my favourites is Polar Co-ordinates and the Cardioid. This curve had some application to Cybernetics in classical (Norbet Wiener's) times. If there is no time for Polars then we can and must instil some sort of intuitive thinking.

Computer software

With the advances made in technology it is now possible to show students graphs of really complicated functions. There is also software to solve equations, calculate derivatives, anti derivatives and areas, volumes and so on. It seems that these "graphing calculators" and similar software are luckily not cheap at this time. My concern is this; ... a computer will solve a "hard" equation or calculate a "hard" definite integral as well as an easy one. I doubt if it is wise to let students depend on it to such an extent that they will need software to solve a quadratic equation or the definite integral of the function x^2 over the interval

$[0, 1]$. Some people call this (dependence) "advanced" or the "new wave" in mathematics teaching. It seems to me (and I am not alone in this) that a more appropriate description would be closer to "mathematics backsliding of the general population" or, some such description which will be offensive to people who think of this as a "new wave". Of course I am not totally against computers/graphing calculators but I question if it is appropriate to (mis) use it to such an alarming extent that students need it to discover the value of the definite integral we mentioned a little earlier, or even for what 7 times 8 is.

§ 3. Critical thinking versus problem solving $(b), (c)$

We have not emphasized in detail this first important part of Mathematics. Instruction as exemplified for example by "epsilon delta". I used to be a fanatical adherent of this topic at students' *very first introduction* to calculus; but now I know better. At this level, why cannot we teach critical thinking by other means? Students who will take Mathematics as their main(or even subsidiary) subject at Universities will see it then. Given the time constraints we teachers deal with, as an alternate, the definite integral as a limit of (Riemann) sums, may be illustrated say, by integrating the function x^2 over the interval $[0\ 1]$.

At the level we are considering, it is more useful to involve students in some project/problem(s) that is not routine. Here is one that a High School student (in a Special Class) asked me in Long Island (NY). I was able to solve it quickly on the phone *only due to the problem solving skills.* I learnt from my excellent teachers at High School in the former

Madras State and also at Madras University. It does NOT need the Calculus but (as you'll see) it needs some geometric insight and some Trigonometry. The angle 60 degrees plays an important part in the solution, as does solving 3 linear equations in 3 variables! Coming back to an over dependence on computers, I cannot imagine how such a student as one at end of \$2 will have the faintest idea how to proceed with this one, much less a correct approach.

"Let $ABCD$ be a square with each side 2 inches. A quarter circle is drawn with center A and radius 2 inches, passing through the vertices B, C and D. A second quarter circle is drawn with B as center and radius 2 inches passing through A, D and C. Two more such circles are drawn with centers at C and D. Find the (exact) area of the curvilinear quadrilateral $PQRS$ formed by intersection of these 4 quarter circles"

Of course all of us Calculus teachers have their own favourite(s); the above is only an example at our level. Why not add yours? By the way, I have tried this one on my second or even 3rd semester Calculus students with some hints. Unfortunately the "hints" became the section! Which to my mind is not what we really should have in mind.

Here is one solution. Well, we want to find the area of $PQRS$. Call it "x". Do we see any other auxiliary areas we can use on the way? There are 4 of them all equal inside the 2 "fish like" regions, 'why not call each as "y"? Likewise call the areas of the remaining four regions z (see the figure).

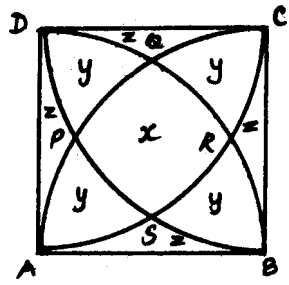

Figure 1.

Next, our plan is to write 3 linear and independent equations in x, y and z. We only want x.

Let us look at the area of the square $ABCD$ which is $2 \times 2 = 4$ sq. inches. This gives our first equation.

$$x + 4y + 4z = 4 \tag{1}$$

Next look at the quarter circle ACB, say. This gives

$$x + 3y + 2z = \frac{1}{4} \cdot \pi \cdot 2^2 = \pi \tag{2}$$

Choice of the third equation needs some care, otherwise you get an equation that is a linear combination of (1) and (2). After some such trials you come up with region AQB whose area $= x + 2y + z$. To find its area we draw QM perpendicular on AB, then curved area $AQB = 2$ (area MQB) $= 2$ (sector $AQB -$ triangle QMB).

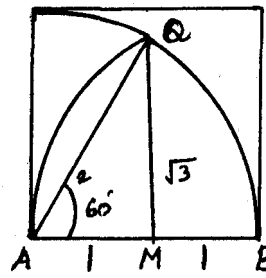

Figure 2.

Looking at right triangle AMQ, we know $AQ = 2$ inches $= AB$, and $AM = 1$ inch; hence angle $\angle QAM = 60° = \frac{\pi}{3}$ and $QM = \sqrt{3}$. We have our third equation, from above, as

$$\begin{aligned} x + 2y + z &= 2\left[\frac{1}{2}2^2 \cdot \frac{\pi}{3} - \frac{1}{2} \cdot 1 \cdot \sqrt{3}\right] \\ &= \frac{4\pi}{3} - \sqrt{3} \end{aligned} \tag{3}$$

Now you eliminate z from equations (1) and (2) to get

$$x + 2y = 2\pi - 4 \tag{4}$$

Then from equations (2) and (3):

$$x + y = \frac{8\pi}{3} - 2\sqrt{3} - \pi = \frac{5\pi}{3} - 2\sqrt{3} \tag{5}$$

Finally eliminating y from equations (4) and (5) we get

$$\begin{aligned} x &= \frac{10\pi}{3} - 4\sqrt{3} - 2\pi + 4 \\ &= \left(\frac{4\pi}{3} - 4\sqrt{3} + 4\right) \\ &= 1.31 \text{ sq. inches, approximately.} \end{aligned}$$

Discussion

(a) There may be other ways to do this; in particular use of calculus to find the last area.

(b) It is interesting that the curvilinear area $PQRS = x$ is only a bit more than 1 which is $\frac{1}{4}$ area of the entire square. Why not compare it with the usual quadrilateral $PQRS$'s area? That is not hard, we need to observe that the line QS is parallel to the sides AD and BC; likewise the line PR to the other 2 sides of the original square.

(c) Just a reminder, "elementary" and "trivial" are not the same.

Conclusion: Teaching Analytical Geometry and Calculus is an important job or service that we teachers provide. We are always on the lookout to improve the quality of instruction as much as we have to pay attention to the quantity.

Non-routine Problems in Mathematics

Lesson 6, Problem 3.

Problem: Prove that the external bisectors of the three angles of a scalene triangle meet their respective opposite sides at three collinear points.

Solution. Let the external bisectors of the angles at A, B, C meet BC, CA, AB at X, Y, Z respectively. Then

$$\frac{BX}{XC} = -\frac{AB}{AC}, \frac{CY}{YA} = -\frac{BC}{AB}, \frac{AZ}{ZB} = -\frac{AC}{BC}.$$

Multiply these, and use the converse of Menelaus theorem.

TWO OBSERVATIONS ON THE TEACHING OF CALCULUS [1]

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1. Matters of notation:

Karl Menger (1902-1985) was one of the influential mathematicians of the twentieth century. He made significant contributions to Dimension Theory, Distance Geometry, Probabilistic Metric spaces, besides Economics, Logic and Pedagogy. Menger was one of the organizers of the famous 'Vienna Circle', a forum for leading mathematicians, scientists and philosophers of his times.

Menger, especially enjoyed teaching students in the evening classes at the Illinois Institute of Technology. This experience led him to write a book on Calculus. Besides other novel features, this book offers a more rational notation / terminology to minimize the difficulties of beginners. In view of the all pervasive nature of mathematics in daily life and the decisive role of Differential and Integral Calculus in the advancement of Science and Technology, Menger's views on the teaching of Calculus are quite significant. In the article [3], Menger notes "nothing is more distasteful to an active mathematician or scientist

[1] Lecture delivered at the One day seminar on Calculus held on 10-11-2001 at I.I.T. (Madras), Chennai.

than discussions of symbolism and notation" and "after having overcome in his youth whatever difficulties the formal expression of idea presents, the mathematician finds that certain ways of writing have become second nature and he regards any suggestion of a change, even if he recognizes its merits, as nothing but a trivial nuisance". He further observes that when "it becomes imperative to make certain techniques and ideas of mathematics available to wider strata of population a thorough discussion of such matters on the highest level is inevitable". Menger illustrates this point by observing that during medieval times banks and laboratories opted for Hindu-Arabic numerals, as the use of Roman numerals for large computations became cumbersome.

Ultimately the use of Hindu-Arabic numerals, proved beneficial to mathematics. With this in view, Menger advocates democratization of mathematical knowledge so that a 'considerable part of the population' learns 'certain techniques of algebra, analytic geometry and calculus as well as some basic ideas of those theories'.

Though Menger wrote the book on Calculus proposing new notation and terminology with this objective in mind, the book itself did not have great impact. Nevertheless, Menger's diagnosis of the difficulties is still both valid and relevant. In support of this, Menger points out how the well-known formula connecting differentiation and integration of a continuous function described in (1.1) below can have both valid and invalid formulations by a slight variation in

notation:

$$\frac{d}{dx}\int_a^x f(x)dx = f(x) \tag{1.1}$$

While, the following formulas

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \tag{1.2}$$

and

$$\frac{d}{du}\int_a^u f(x)dx = f(u) \tag{1.3}$$

mean the same thing as (1.1), formulas such as

$$\frac{d}{du}\int_a^x f(x)du = f(x) \tag{1.4}$$

and

$$\frac{d}{du}\int_a^u f(x)dx = f(u) \tag{1.5}$$

are in general false. Menger's article [2] "Why Johnny hates math?" is a further elaboration of similar notational / terminological difficulties faced by beginners. Writing $\frac{dy}{dx} = \frac{y}{x}$ after cancelling d is another mistake committed by a few students even to-day. Certainly, there, is thus a need to caution the beginning students against such notational misadventures. Though this may seem trivial for a trained student/ teacher, teachers should spend some time clarifying this aspect.

2. A basic limit in Calculus

My second observation is on the proof of the result: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. The standard proof makes use of the formula for the area of a sector of a circle (see Proof of Theorem 3 in

the article by Dr. G.P.Youvaraj in this issue of Mathematics Teacher). Following Hoffman [1] one can altogether avoid considerations of area by relying on estimates of arc length. In a way, this proof is in the spirit of Menger , as it deals with the problem in a 'one dimensional' rather than 'two dimensional' setting. A proof due to Perry [4] also avoids the use of area but is based on integral calculus and mean value theorem.

First Alternative proof (see Hoffman[1])

The following result called "Sandwich Principle" or "Squeeze Lemma" is well-known:

Theorem 1. Let $f, g, h : (a,b)(\leq \mathbf{R}) \to \mathbf{R}$ be functions where $a < b$; if $x_0 \in [a,b]$ and $f(x) \leq g(x) \leq h(x)$ for all x in (a,b) and if $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x)$ exists and equals L.

Lemma 2. $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta} = 0.$

Proof. In the unit circle with centre $O = (0,0), P$ be an arbitrary point $(\cos\theta, \sin\theta)$ and let $P' = (\cos\theta, 0)$ be the projection of P on the $X-$axis. Let A be $(1,0)$ and Q the point $(1, \sin\theta)$ see Figure 1).

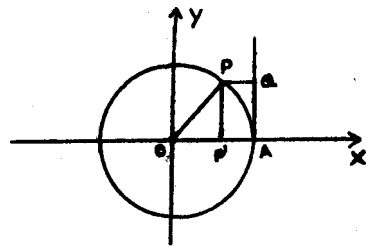

Figure 1.

The length of the line segment PA is less than the length of arc joining P and A. The arc length PA is θ (in radian measure). Thus for $\theta > 0$ and sufficiently small, $0 < \theta < \frac{\phi}{2}$ and $0 \leq PA \leq$ are length PA or

$0 \leq \sqrt{(1-\cos\theta)^2 + \sin^2\theta} \leq \theta$; i.e. $0 \leq 2(1-\cos\theta) \leq \theta^2$, or $0 \leq (1-\cos\theta)/\theta \leq \theta/2$.

If $\theta < 0$ and small, $\frac{\theta}{2} \leq \frac{1-\cos\theta}{\theta} \leq 0$.

So, by Theorem 1, $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$.

Theorem 2. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Proof. In the above figure for $0 < \theta < \frac{\pi}{2}$, it is clear that [2]

$PP' \leq ArcPA \leq PQ+QA$. So, $\sin\theta \leq \theta \leq 1-\cos\theta+\sin\theta$. In other words

$$\theta \leq 1 - \cos\theta + \sin\theta \leq 1 - \cos\theta + \theta.$$

Dividing by θ throughout we get

$$1 \leq \frac{1-\cos\theta}{\theta} + \frac{\sin\theta}{\theta} \leq \frac{1-\cos\theta}{\theta} + 1.$$

Now by Lemma 2 and Sandwich principle it follows that $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

One can develop first Integral Calculus and then at the appropriate stage offer the definition of derivative of a

[2] This intuitively clear inequality is preferred here to the assumption that area of triangle OAP is less than that of the sector OAP.

function to be followed by the statement and demonstration of the basic properties of derivatives. In this scheme of topics, the definition of definite integral as a limit of Riemann sums / upper and lower sums and the mean - value theorem of Integral Calculus play a definitive role. Following this sequence of ideas we can offer an alternative proof of Theorem 2 above essentially due to Perry [4].

We state below

Theorem 3 (Mean-Value Theorem of Integral Calculus)

If $f : [a,b](a < b) \to \mathbf{R}$ is continuous then there exists $x_0 \in [a,b]$ such that $f(x_0) = \frac{1}{(b-a)} \int_a^b f(x)dx$.

Second Alternative proof (see Perry [4])

Let $P = (x,y) = (\cos\theta, \sin\theta)$ be a point on the unit circle in the first quadrant. Then by the formula for arc length

$$\theta = \int_0^y \frac{du}{\sqrt{1-u^2}} (0 \leq y < 1) \tag{2.1}$$

The derivation of this relation can be obtained from first principles in the same way we obtain the formula for length of arc by considering the distance between contiguous points (x,y) and $(x+\Delta x, y+\Delta y), y$ being $\sqrt{1-x^2}$.

Since $y = \sin\theta, by (2.1)$

$$\frac{\theta}{\sin\theta} = \frac{1}{y} \int_0^y \frac{du}{\sqrt{1-u^2}} \tag{2.2}$$

In view of Theorem 3, (2.2) can be written as

$$\frac{\theta}{\sin\theta} = \frac{1}{\sqrt{1-\theta_1^2}}, \tag{2.3}$$

where $0 \leq \theta_1 \leq \sin\theta < \theta$. Allowing θ to tend to 0 in (2.3), we obtain

$$\lim_{\theta\to 0} R.H.S \text{ of } (2.3) = \lim_{\theta\to 0} \frac{1}{\sqrt{1-\theta_1^2}} = 1$$

So

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1.$$

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SOME CONCEPTUAL DIFFICULTIES IN LEARNING CALCULUS[1]

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While learning Calculus at + 2 level, in general, students find two types of difficulties - (1) Conceptual and (2) Computational. Conceptual difficulties are different from computational ones in that a resolution of this type of difficulties in essential for a clear understanding of the basis of Calculus. In what follows we will try to list out some of the conceptual difficulties in learning Calculus.

Limits

(1) Students find it difficult to distinguish between the expressions $\lim_{n\to\infty} x^n$ and $\lim_{x\to a} x^n$. $x^n \to 0$ as $n \to \infty$ through the values $1, 2, 3, \cdots$ when $|x| < 1$, $x^n \to \infty$ when $x > 1$. x^n oscillates between $-\infty$ and ∞ as $n \to \infty$ when $x < -1$. When $x = -1, x^n$ oscillates between -1 and $+1$ as $n \to \infty$. When $x = 1$, $\lim_{n\to\infty} x^n = 1$. On the other hand, $x^n \to a^n$ as $x \to a$ when a is any fixed real number.

(2) Teachers generally start teaching the concept of limit with examples like $\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$ (of the form $\frac{0}{0}$). Here students find difficulty in distinguishing between $\lim_{x\to a} f(x)$

[1] Lecture delivered at the One day seminar on Calculus held on 10-11-2001 at I.I.T. (Madras), Chennai.

and $f(a)$. The difficulty can be resolved by preferably giving several examples of functions when $\lim_{x \to a} f(x)$ is not equal to $f(a)$. As illustrations let us have the following:

$$\text{(a)If} \quad \left.\begin{array}{rcl} f(x) & = & x+2 \text{ for } x \neq 2 \\ & = & -2 \text{ for } x = 2 \end{array}\right\}, \quad \text{find} \quad \lim_{x \to 2} f(x)$$

and $f(2)$.

Here $\lim_{x \to 2} f(x) = 2$, but $f(2) = -2$.

The graph of $f(x)$ can be drawn as follows:

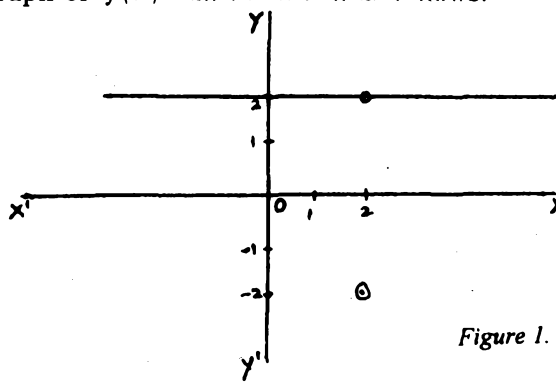


Figure 1.

Here a blank circle e.g., ○ on the graph means that the graph does not run through the point, An encircled thick dot e.g. ⊙ indicates that the point lies on the graph.

(b) If $f(x) = \frac{x^2-9}{x-3}$ for $x \neq 3$ and $f(3) = 10$, find $\lim_{x \to 3} f(x)$.

Here $\lim_{x \to 3} f(x) = 6$ and $f(3) = 10$.

3. Students also face difficulty in understanding the full import of the following definition of the limit $lim_{x \to a}(x)$. $\lim_{x \to a} f(x) = \lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$. Here for the resolution of this difficulty the teacher should explain clearly (with the help of graphical illustration of the function

and their limits), what phrases like 'left hand limit', 'right hand limit' and 'the value of the function at $x = a$', 'x approaching a through values less than a' and 'x approaching a through values of x greater than a'. For example consider

$$\begin{aligned} f(x) &= x^2 \text{ for } x > 2 \\ &= 0 \text{ for } x = 2 \\ &= -x^2 \text{ for } x < 2 \end{aligned}$$

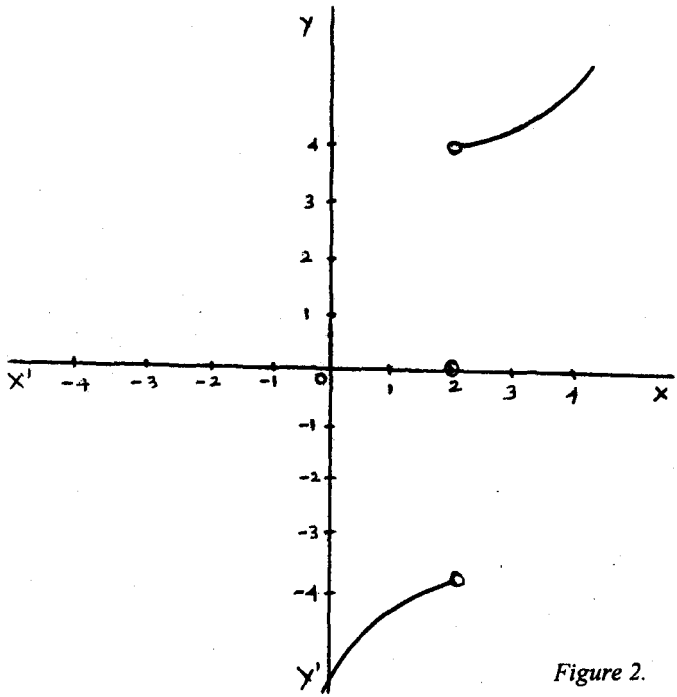


Figure 2.

Here the right hand limit $\lim_{x \to 2+} f(x) = 2^2 = 4$, the value of the function $f(x)$ at $x = 2$ is $f(2) = 0$ and the left hand limit $\lim_{x \to 2-} f(x) = -2^2 = -4$. As here $\lim_{x \to 2+} f(x) \neq f(2) \neq \lim_{2-} f(x)$, $\lim_{x \to 2} f(x)$ does not exist.

(4) $\epsilon - \delta$ definition of limit is one of the most difficult concepts for the students. But at the same time it is an essential tool in higher mathematics, So, the teacher should be careful in choosing the time and the students when and to whom to teach this concept. If in any course of Calculus it is possible to avoid this concept and proceed to the applications of Calculus without it, it is better to avoid the $\in -\delta$ definition of limit (Ref. 1 and Ref. 2).

Continuity

(1) The concept of continuity of a function at a point is intimately connected with that of limit. The teacher can safely start with the following definition of continuity, and not with its $\epsilon - \delta$ definition:-

Definition. A function $f(x)$ is said to be continuous at the point $x = a$ if and only if $\lim_{x \to a+} f(x) = f(a) = \lim_{x \to a-} f(x)$ in the sense that both the limits should exist and be equal to $f(a)$.

This concept can be illustrated with several examples of whether a given function is continuous at a given point, By doing so the teacher can bring out clearly the difference between continuity and discontinuity of a function at a point. For example, whereas $f(x) = x + 2$ is continuous at $x = 2, f(x) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$ if $f(2) = 5$.

At this juncture the teacher can distinguish between a local property and a global property while teaching the continuity of a function over an interval. The definition

of continuity of a function at the different end points of an interval may be brought out clearly.

(2) The teacher should use several examples of functions and their graphs to explain the different types of discontinuity. This will help the student to draw the graphs of functions correctly. Types of discontinuity are given below with examples (Ref.3,pp.188-189).

(a) Let $f(a+0) \neq f(a-0)$. Here $f(a)$ may be equal to one or to neither of $f(a+0)$ and $f(a-0)$ or may be undefined. For example, for $f(x) = [x]$, integral part of $x, f(0-0) = -1$, but $f(0+0) = f(0) = 0$. Also, for $g(x) = [x] - [-x], g(0-0) = -1, g(0+0) = 1, g(0) = 0$. And, for $h(x) = [x] + x \sin \frac{1}{x}, h(0-0) = -1, h(0+0) = 0$ and $h(0)$ is undefined. In these cases, $f(x), g(x)$ and $h(x)$ have a discontinuity at $x = 0$.

(b) It may happen that $f(a+0) = f(a-0)$, but $f(a)$ is not defined or is different from $f(a+0)$ and $f(a-0)$. For example, for $f(x) = x \sin \frac{1}{x}$. When $x = 0$, we have $f(0+0) = f(0-0) = 0$ and $f(0)$ is not defined. Also, for $h(x) = [1-x^2]$, when $a = 0$, we have $h(0+0) = h(0-0) = 0$ and $h(0) = 1$.

The type of discontinuity mentioned in (a) and (b) is generally called a simple discontinuity To these cases we may also add those cases wherein only one of $f(a+0)$ or $f(a-0)$ exists, but $f(a)$ is not defined or is not equal to $f(a+0)$ or $f(a-0)$.

(2) $f(a+0)$ or $f(a-0)$ or both may be $+\infty$ or $-\infty$. For example $f(x) = \frac{1}{x}$ at $x = 0$ and $g(x) = \tan x$ at $x = \frac{\pi}{2}$. Such discontinuity is called an **infinite discontinuity**.

(3)Any point of discontinuity which is neither a simple one nor an infinite one is called a point of **oscillatory** discontinuity. For example, $f(x) = \sin \frac{1}{x}$ has such a discontinuity at $x = 0$.

Derivative

The concept of derivative appears so much unfamiliar to the students in the beginning that they start doubting the importance of this concept. So the teaching of derivative should start with copious illustrations from Physics or Economics e.g, velocity, acceleration and marginal utility. The teacher should relate these concepts from the applied subjects with that of $\frac{dy}{dx}$ or $\frac{dx}{dt}$, taking into consideration the general back ground (e.g. science/commerce) of the class.

Before teaching $\frac{dy}{dx}$ as $\lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$ or $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$, the teacher should explain the difference between "average rate of change during the small (infinitesimal) interval δx and "the instantaneous rate of change at x (i-e, at the moment/at the point)". The teacher should stress the point that the instantaneous rate is the limit of the average rate as the interval (δx) becomes smaller and smaller (i.e., tends to 0).

(2) While teaching the above concept the teacher should explain clearly that $\frac{dy}{dx} \neq \frac{y}{x}$, even though $\frac{dy}{dx} = \frac{1}{(dx/dy)}$.

where $\frac{dx}{dy} = \lim_{\delta y \to 0} \frac{\delta x}{\delta y}$.

The teacher should explain the difference between dx and δx or dy and δy. Whereas dx and dy are called differentials, δx and δy are infinitesimal (very small) increments in x and y respectively. The difference between these may be made clear through the following graph and example (Ref.2,p.178). Here $\delta x = PR$ and $\delta y = RQ$

$$\begin{aligned} dy &= RT = \tan\alpha . PR \\ &= f^1(x)\delta x \\ &= \frac{dy}{dx} \cdot \delta x \end{aligned}$$

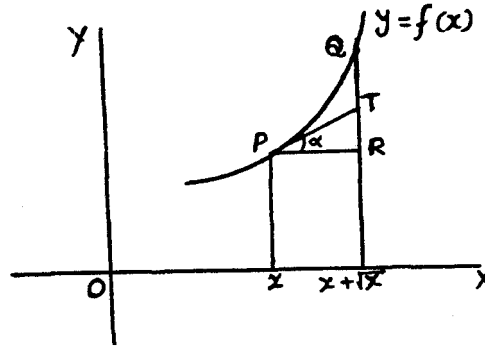


Figure 3.

Example. For the function $y = x^2$, as x changes from 3 to 3.1. the increment $\delta y = (x+\delta x)^2 - x^2 = 2x \cdot \delta x + (\delta x)^2 = 2.3 \cdot (0.1)^2 + (0.1)^2 = 0.61$. The differential $dy = \frac{dx^2}{dx}\delta x = 2x\delta x = 2.3 \cdot (0.1) = 0.6$.

Taking dy as an approximate value of δy, we have the absolute error in approximation $= \delta y - dy = 0.01$ and the relative error in approximation $= \frac{\delta y - dy}{dy} = \frac{0.01}{0.60} = \frac{1}{60} = 1.7\%$

A bit of history of Calculus may be brought in while

explaining the concept of differentials and their importance. (Ref.4 pp 433-435,and Ref.5,pp.331-340)

(3) Students find it easy to understand the import of $\frac{dy}{dx}$ and $\frac{dx}{dt}$, where 't' denotes the time variable. But sometimes they find it relatively difficult to understand the phrase like "the rate of change of area $(= x^2)$ of a square with respect to the side x. Sometimes it is helpful to connect the variables x, x^2 with time variable just to explain the meaning of the process involved.

Time t in seconds	1	2	3	4	...	...	...	...
Side x in cm.	2	3	5	7	...	...	...	...
Area x^2 in sq.cm	4	9	25	49	...	...	...	...

This scheme of teaching can help also in the explanation of the formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

(4) While teaching the geometrical interpretation of $\frac{dy}{dx}$ the teacher should explain the relationship between increasing graph, decreasing graph on one side and $\frac{dy}{dx}$ on the other. He may take examples of functions like $y = x^{2/3}$ or $y = 1/x^2$ for explaining there.

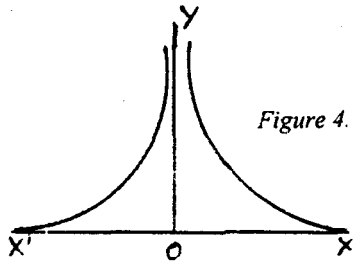


Figure 4.

The graph of $y = \frac{1}{x^2}$

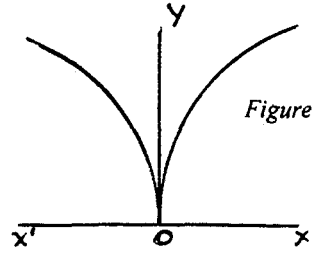


Figure 5

The graph of $y = x^{2/3}$

Here he may also explain the significance of $f'(x) > 0, f'(x) < 0$ and $f'(x)$ by drawing suitable diagrams (Ref,6 pp57) This will clarify the relationship between $\frac{dy}{dx}$ and slope of a curve at a point.

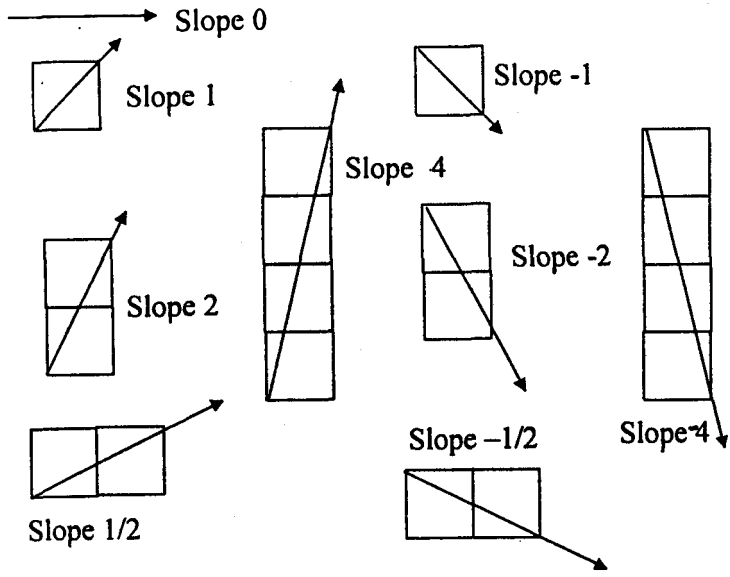


Figure 6.

(5) While teaching the conditions of maxima and minima the teacher should discuss the graphs of $f(x), f'(x), f''(x)$ and explain that though graphs of $f'(x)$ and $f''(x)$ give information about the function $f(x)$, they in themselves are functions different from $f(x)$. At this point the teacher may relate the differentiability and continuity to emphasise that a function differentiable at a point is continuous at the point, but the converse need not be true. He may give examples of functions, involving $\sin \frac{1}{x}$ etc., but the best and easiest example is $|x|$, as is shown below:

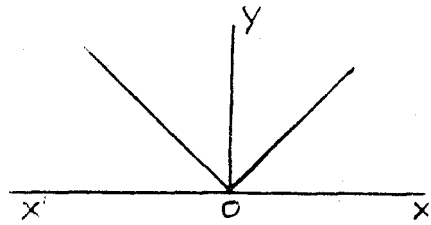


Figure 7.

Graph of $y = f(x) = |x|$

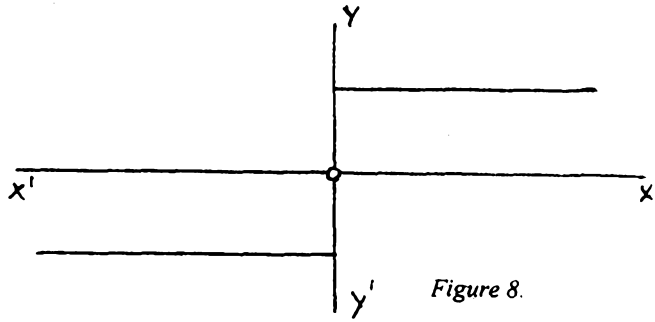


Figure 8.

Graph of $y = f'(x)$ where $f(x) = |x|$

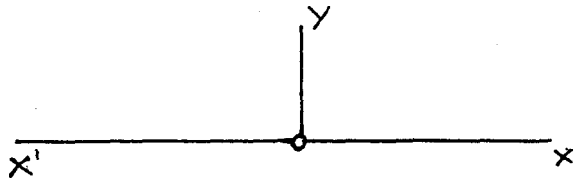


Figure 9.

Graph of $y = f''(x)$ where $f(x) = |x|$

The teacher may take the help of elementary functions like $\sin x$ and $\cos x$ and their graphs to explain the relationship of critical points (points of maxima, minima or inflexion) with the increasing or decreasing of the functions.

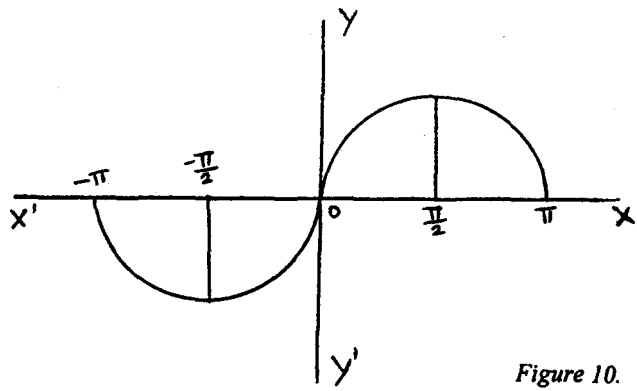


Figure 10.

Graph of $y = \sin x$.

In the above figure the graph of $y = \sin x$ in $[-\pi, \pi]$ is considered. The function increases with x in $[-\pi/2, \pi/2]$ and decreases with x in the intervals $[-\pi, \frac{-\pi}{2}]$ and $[\frac{\pi}{2}, \pi]$. Points $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are points of maxima and minima for the function $y = \sin x$.

It should also be made clear that vanishing of the derivative is only necessary for the existence of extremum.

Integration

Students are generally confused while correlating the definitions of $\int f(x)dx$ as a limit of an infinite sum and as the anti derivative of $f(x)$. In order to resolve this confusion it is better to teach the Fundamental Theorem of Integral Calculus as early as possible in the teaching schedule and give an intuitive proof of the same (Ref.3.pp.268-269). One such proof is given below.

Suppose that P_0PP' is the graph of a continuous function $y = f(x)$ which lies wholly above the $x-$axis. Let P and P' be the points (x, y) and $(x+h, y+k)$, where h may be positive or negative (positive in the figure).

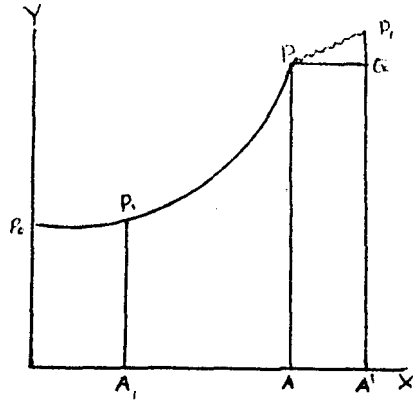


Figure 11.

Our problem is to calculate the area $OAPP_0$. Let this area Π which is a positive area be denoted by a function $F(x)$. Let $(OAPP_0)$ denote the area of the region $OAPP_0$ and similarly for other regions. Then clearly from the figure, we have $OA = x, AA' = h, PA = f(x)$. Then $(AA'QP) = hf(x), (PQP') < hk < |h|\lambda(h)$ where $\lambda(h)$ is the greatest distance of any point of the arc PP' from PA. Also,

$$(PQP') + (AA'QP) = (AA'P'P) \text{ and } (A_1APP_1) < (OAPP_0)$$

Now, $$\begin{aligned}(AA'P'P) &= (OA'P'P_0) - (OAPP_0)\\ &= F(x+h) - F(x)\\ &= (AA'QP) + (PQP')\\ &= h\,[f(x) + \mu(h)]\end{aligned}$$

where $\mu(h) \leq \lambda(h)$ and $\lambda(h) \to 0$ as $h \to 0$.

Therefore $$\frac{F(x+h) - F(x)}{h} = F(x)\mu(h)$$

Hence $$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}[f(x) + \mu(h)]$$

$$\text{or} \qquad F'(x) = f(x)$$

$$\text{or} \qquad \frac{dF(x)}{dx} = f(x) \Rightarrow F(x) = \int f(x)dx$$

where $\int f(x).dx$ is the antiderivative of $f(x)$ Thus we have proved the following theorem.

Theorem. The ordinate of the curve $f(x)$ is the derivative of the area $F(x)$, and the area $F(x)$ is the integral of the ordinate.

More formally, the function $F(x) = \int_a^x f(t).dt$ has a derivative equal to $f(x)$.

Note: Even if the graph of $f(x)$ runs below the x-axis, the Fundamental Theorem is valid and suitable modifications can be made in the proof. Also, it does not matter where the point A is situated on the x-axis, as $\frac{dy}{dx}$ does not change on this account.

Some Teaching Aids

For clear understanding of the relationship between the derivative and the integral, the teacher can illustrate geometrically with the help of diagrams and 3-dimensional models as given below (Ref.6,PP.88-92, and Ref.7.pp.52-94). There aids give us a distinct feel of what is a derivative and what is an integral,

(a) $\frac{dx^2}{dx} = 2x.$

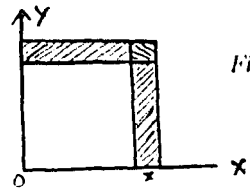


Figure 12.

In the above figure a square is drawn with side x. Then another square is drawn with side $x + \delta x$. (i.e.,after giving an increment δx to x). Then the change δA in the area A of the square is denoted in the figure by two shaded strips of width δx on two sides of the square. As seen clearly from the figure, the increment δA is given by

$$\begin{aligned}
\delta A &= A(x+\delta x) - A(x) = (x+\delta x)^2 - x^2 \\
&= x^2 + 2x \cdot \delta x + \delta x^2 - x^2 \\
&= 2x \cdot \delta x + \delta x^2 \\
\Rightarrow \frac{\delta A}{\delta x} &= 2x + \delta x \Rightarrow \lim_{\delta x \to 0} \frac{\delta A}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) \\
\Rightarrow \frac{dA}{dx} &= 2x.
\end{aligned}$$

In the figure, as δx becomes indefinitely small the two strips approach the two sides of length x, So in the limit the rate of change of A w.r.to x is represented by the sum of the lengths of 2 sides i.e., $2x$. That is, $\frac{dA}{dx} = 2x$.

The integral $\int_0^x 2x \cdot dx = x^2$ can be represented by the following figure.

In this figure the length and breadth of a square with area x^2 has been divided into n parts each of width $\frac{x}{n}$. Then the area of the square is calculated as

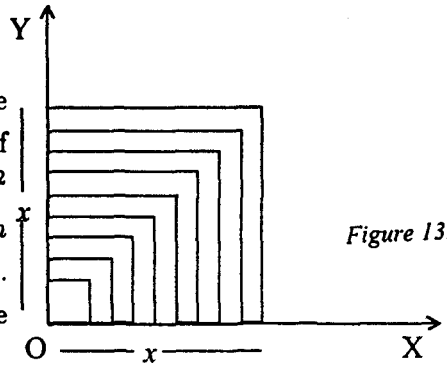


Figure 13.

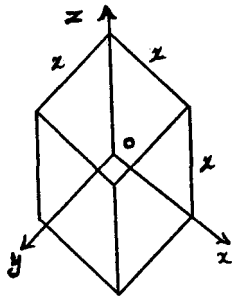
the limit of the sum of these strips as $n \to \infty$. The limit of the sum is found to be x^2.

In fact, we have,

$$A = \lim_{n\to\infty} \sum_{r=1}^{n} 2 \cdot \frac{rx}{n} \cdot \frac{x}{n} = \lim_{n\to\infty} \frac{2x^2}{n} \sum_{1}^{n} r$$

$$= \lim_{n\to\infty} \frac{2x^2}{n^2} \cdot \frac{n(n+1)}{2} = x^2.$$

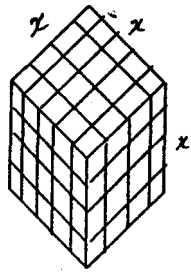
(b) As in (a), the following 3-dimensional figures illustrate (i) $\frac{dx^3}{dx} = 3x^2$ and (ii) $\int_0^x 3x^2 \cdot dx = x^3$



$\frac{dx^3}{dx} = 3x^2$

Figure 14.

Here 3 faces each of area x^2 together give the derivative of x^3 as $3x^2$.



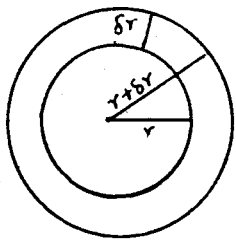
$\int_0^x 3x^2 \cdot dx = x^3$

Figure 15.

A cube of side x

(c) Similarly, we can show that the derivative of the volume $\frac{4\pi r^3}{3}$ of a sphere with respect to its radius r is given by its surface area $4\pi r^2$.

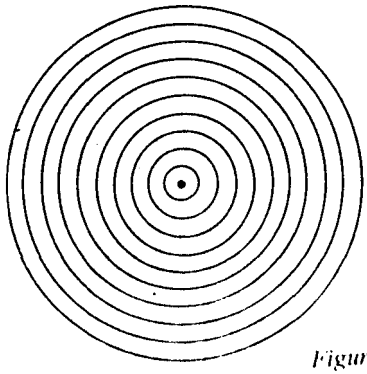
Also by dividing a sphere into any number of concentric spherical shells of equal width and taking the limit of the sum of volumes of these spherical shells (as the width of each shell approaches zero), we get the volume of a sphere of radius r as $\frac{4\pi r^3}{3}$



$$\frac{d\left(\frac{4\pi r^3}{3}\right)}{dr} = 4\pi r^2$$

Figure 16.

A shell of inner radius r and width δr.



$$\int_0^r 4\pi r^2 = \frac{4\pi r^3}{3}$$

Figure 17

A sphere divided into concentric shells of equal width.

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Non-routine Problems in Mathematics

Lesson 5, Problem 4.

Problem: Find all positive integers which can be expressed as the sum of three distinct composite numbers.

Solution. If n is the sum of three distinct composite numbers, then $n \geq 4 + 6 + 8 = 18$. Let $n \geq 18$. If n is even then $n - 10$ is an even number greater than 6, and $n = 4 + 6 + (n - 10)$. If n is odd then $n - 13$ is an even number greater than 4, and $n = 4 + 9 + (n - 13)$. So the desired positive integers are all $n \geq 18$.

SOME IMPORTANT LIMITS IN CALCULUS[1]

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Abstract

The concept of limit is probably second only to the concept of function. Without the concept of limit, about a half (may be more) of Mathematics that exists today will not exist. It is so fundamental that every student of Mathematics needs to understand it clearly in order to pursue Mathematics or its applications. Also, we shall consider some important limit ideas in Calculus and their applications. Also, we shall give web resources that will be useful, as a tool, in teaching these ideas.

We shall only consider real-valued functions that are defined on a subset of $\mathbf{R}$. What is the limit of a function? Little or more precisely, what is the limiting value of the function values $f(x)$ as x approaches a point say $x = b$? This is an interesting and important question; because in most of the applications we want to know how the dependent variable changes as its independent variable is changed.

Definition. Let f be a real-valued function that is defined on an open interval $(a, b) \subseteq \mathbf{R}$ containing point c, except

[1] Lecture delivered at the One day seminar on Calculus held on 10-11-2001 at I.I.T. (Madras), Chennai.

possibly at the point c itself. We say that $f(x)$ has a limit value L as x approaches c, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in (a, b)$ such that $0 < |x - c| < \delta$. When such a limit exists, we denote it by writing $\lim_{x \to c} f(x) = L$.

What do these, ϵ, δ mean? Given a tolerance level $\epsilon > 0$, there exists a $\delta > 0$ that guarantees that $f(x)$ values are within ϵ distance from L for all those $x \in (a, b)$, except possibly for c, that are within the δ distance from c. If this is possible for every ϵ, then the limit exists and we write $\lim_{x \to c} f(x) = L$. Even if it fails for one $\epsilon > 0$, then $\lim_{x \to c} f(x)$ **does not exist**. That is, when $f(x)$ has a limit L the values of $f(x)$ gets closer to L as x in (a, b) gets closer to the point c.

Note that x values may approach c from either side. That is,

$$\lim_{x \to c} f(x) = L \text{ if and only if } \lim_{x \to c^+} f(x) = L \text{ and } \lim_{x \to c^-} f(x) = L$$

where $\lim_{x \to c^+}$ means $x > c$ and approaches c (i.e., from the right hand side of c); and $\lim_{x \to c^-}$ means $x < c$ and approaches c (i.e., from the left hand side of c). We call these limits as right hand, left hand limits respectively.

Remarks. (1) One important observation that we need to make is that, in order to talk about the limit of f at c, it is not necessary that the function be defined at c. It is enough that f is defined in a neighbourhood (an open interval containing c) of c.

(2) Function f has a limit at c if and only if f has both

right and left hand limits and they are equal.

(3) When our $c = b$, the right end point of our interval (a, b), then we understand the $\lim_{x \to b} f(x) = L$ as $\lim_{x \to b^-} f(x) = L$ and similarly if $c = a$, then we understand $\lim_{x \to a} f(x) = L$ as $\lim_{x \to a^+} f(x) = L$.

(4) Since every sequence can be considered as a function defined on $\mathbf{N}$, the set of all positive integers, we can talk about the limit of a sequence as $n \to \infty$ in the same manner. That is, a sequence $\{a_n\}_1^\infty$ is said to have a limit L, if for every $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that $|a_n - L| < \epsilon$ for all $n > N$. When such a limit exists we say that the sequence $\{a_n\}_1^\infty$ converges to L.

(5) If a function f has a limit L as $x \to c$, then L is unique. Similarly, limit of a sequence is also unique.

Theorem 1. *Let f, g be two real-valued functions defined on an open interval I containing c. If $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then the following hold:*

(i) $\lim_{x \to c} (f \pm g)(x) = L \pm M$

(ii) $\lim_{x \to c} (fg)(x) = LM$

(iii) $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$, *provided* $M \neq 0$.

Proof is easy and omitted.

The following theorem, also known as "Squeezing Principle", is very useful in the limiting process.

Theorem 2. *Let f, g and h be real-valued functions defined on an open interval I containing c, except possibly at c itself, and there exists a $\delta_0 > 0$ satisfying*

$$g(x) \leq f(x) \leq h(x) \text{ for } x \in I \text{ such that } 0 < |x - c| < \delta_0$$

and $\lim_{x \to c} g(x) = L$, *and* $\lim_{x \to c} h(x) = L$, *Then* $\lim_{x \to c} f(x) = L$.

Proof. Let f, g, h, c, δ_0, and L be as in the hypothesis. Let $\epsilon > 0$ be given. Since $\lim_{x \to c} g(x) = L$, we can find a $0 < \delta_1 < \delta_0$ such that $|g(x) - L| < \epsilon$ for all $x \in I$ such that $0 < |x - c| < \delta_1$. Also, $\lim_{x \to c} h(x) = L$ so we can find a $0 < \delta_2 < \delta_0$ such that $|h(x) - L| < \epsilon$ for all $x \in I$ such that $0 < |x - c| < \delta_2$. Now choose $\delta = \min\{\delta_1, \delta_2\}$. Then for x such that $0 < |x - c| < \delta$ we have

$$|f(x) - L| \leq \max\{|g(x) - L|, |h(x) - L|\} < \epsilon$$

which completes the proof.

Example. Let $f(x) = x\cos(1/x)$ for $x \neq 0$. Note that $f(x)$ is not defined at $x = 0$. Since $|\cos(1/x)| \leq 1$ for all $x \neq 0$, we have $-|x| \leq |x\cos(1/x)| \leq |x|$ for all $x \neq 0$. Now by Theorem 2 it follows that $\lim_{x \to 0} x\cos(1/x) = 0$.

The following theorem is one of the most important limits in Mathematics. We shall use Theorem 2 to prove it.

Theorem 3. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

Proof. First we shall prove that $\lim_{x \to 0^+} \dfrac{\sin x}{x} = 1$. Suppose that $0 < x < \pi/2$. Consider the figure given below,

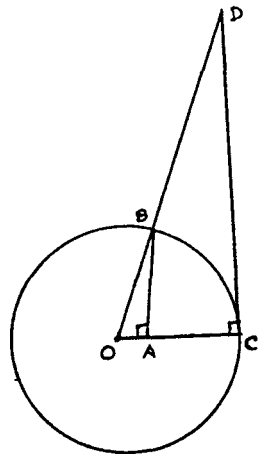

Figure 1.

where the circle is the unit circle and AB and CD are perpendicular to the radius OC. If x is the arc length, then in $\triangle OAB, AB = \sin(x)$, and $OA = \cos(x)$. Note that $\triangle OAB, \triangle OCD$ are similar; hence $CD = \tan(x)$. Now, from the figure,

Area of $\triangle OAB <$ Area of the sector $OCB <$ Area of $\triangle OCD$ [2].

Thus $\frac{1}{2}\cos(x)\sin(x) < \frac{x}{2} < \frac{1}{2}\tan(x)$, which implies

$$\cos(x) < \frac{x}{\sin(x)} < \frac{1}{\cos(x)} . \tag{1}$$

Now from the figure, note that as $x \to 0^+, OA \to OC = 1$, and hence $\lim_{x\to 0^+} \cos(x) = 1$. Then by (1) and Theorem 2, we have $\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$. For $x < 0$, note that $\frac{\sin(-x)}{-x} = \frac{\sin(x)}{x}$ and hence $\lim_{x\to 0^-} \frac{\sin(x)}{x} = 1$ follows from what we have proved.

[2] For an alternative treatment see the proof of Lemma 2 in the article by Prof. P.V.Subrahmanyam in this issue.

This completes the proof. ■

Observation. While proving Theorem 3 we have also proved $\lim_{x\to 0} \cos(x) = 1$. Also, from the figure above it follows that $\lim_{x\to 0} \sin(x) = 0$.

Corollary.
(i) $\lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$, (ii) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$.

Proof. Consider

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{1+\cos(x)}$$
$$= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} = 1 \cdot 0 = 0$$

using Theorem 3 and Theorem 1 (iii).

Now consider the other limit

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin^2(x)}{x^2(1+\cos(x))}$$
$$= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 \cdot \frac{1}{1+\cos(x)}$$
$$= \lim_{x\to 0} \left(\frac{\sin(x)}{x}\right)^2 \cdot \lim_{x\to 0} \frac{1}{1+\cos(x)} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

again using Theorem 3 and Theorem 1 (iii). ■

Another important concept in Calculus, which is related to "limit", is continuity of a function at a point.

Definition. A function g defined on $S \subset \mathbf{R}$ is said to be continuous at a point $b \in S$ with a neigbhourhood of b contained in S if the following holds:

(i) g is defined at the point $x = b$

(ii) $\lim_{x \to b} g(x) = L$ exists

(iii) $g(b) = L$

Even if one of the above three conditions fails, then the function is not continuous. Continuity is also equivalently stated as

"g is continuous at $x = b$ if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that $|g(x) - g(b)| < \epsilon$ whenever $x \in S$ and $|x - b| < \delta$".

Theorem 3 is useful in finding the derivatives of $\sin(x), \cos(x)$; and in the calculation of $\int_a^b \sin(x)dx$ and $\int_a^b \cos(x)dx$.

Theorem 4.

(i) $\frac{d}{dx}\sin(x) = \cos(x)$ *for all* $x \in \mathbf{R}$

(ii) $\frac{d}{dx}\cos(x) = -\sin(x)$ *for all* $x \in \mathbf{R}$

(iii) $\int_a^b \sin(x)dx = \cos(a) - \cos(b)$

(iv) $\int_a^b \cos(x)dx = \sin(b) - \sin(a)$.

Proof: We shall prove only (i) and (iii) as the proofs (ii) and (iv) are similar to (i) and (iii) respectively. Consider

$$\begin{aligned} \frac{d}{dx}\sin(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \end{aligned}$$

$$= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h}$$
$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

where we have used Corollary 1 (i) and Theorem 3. This proves (i).

To prove (iii), let $n \in \mathbf{N}$, and $h = \frac{b-a}{n}$. Then $h \to 0$ as $n \to \infty$. Consider the partition, $x_0 = a, x_1 = a+h, x_2 = a+2h, \cdots x_k = a+kh, \cdots b = a+nh$. Then

$$\begin{aligned}
\int_a^b \sin(x)dx &= \lim_{n\to\infty}[\sin(a+h)+\cdots+\sin(a+kh)+\cdots \\
&\quad +\sin(a+nh)]h \\
&= \lim_{n\to\infty} \frac{h}{2\sin(h/2)} \sum_{k=1}^{n} 2\sin(a+kh)\sin(h/2) \\
&= \lim_{n\to\infty} \frac{h}{2\sin(h/2)} \sum_{k=1}^{n} [\cos(a+(k+1/2)h) \\
&\quad -\cos(a+(k+1/2)h)] \\
&= \lim_{n\to\infty} \frac{h}{2\sin(h/2)} [\cos(a-h/2) \\
&\quad -\cos(b+h/2)] = 1\cdot[\cos(a)-\cos(b)].
\end{aligned}$$

Note that Theorem 3, and the continuity property of $\cos(x)$ are (crucial) used in the last step in taking limit. ■

Here is another application of Theorem 2.

Theorem 5. *Suppose that f is Riemann integrable over $[a,b]$. Then the function $A(x) = \int_a^x f(t)dt, x \in (a,b)$, is continuous on (a,b).*

Proof. Let us choose and fix an arbitrary point $x_0 \in (a,b)$.

It is enough to show that A is continuous at x_0. Since f is integrable over $[a,b]$, there exists a constant $M > 0$ such that $|f(t)| \leq M$ for all $t \in [a,b]$. Then

$$-M|x - x_0| \leq A(x) - A(x_0) = \int_{x_0}^{x} f(t)dt \leq M|x - x_0|.$$

Thus for a given $\epsilon > 0$ we can choose $\delta = \frac{\epsilon}{M}$, to see that $A(x)$ is continuous at x_0.

This completes the proof.

In fact, if $f(x)$ in Theorem 5 is continuous on (a,b), then it can be shown, using Theorem 2, that $A(x)$ is differentiable on (a,b). The function $A(x)$ defined in Theorem 5 is called the area function for f. Next we shall turn our attention to look at some important sequential limits in Calculus. The following lemma is needed in the sequel.

Lemma 1. *For any $h > 0$, and $n \in \mathbf{N}$ we have*

$$\text{(i) } (1+h)^n > 1 + nh, \qquad \text{(ii) } (1+h)^n > \tfrac{n(n-1)}{2}h^2.$$

Proof: Both (i) and (ii) follow by the binomial theorem

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \frac{n(n-1)(n-2)}{3}h^3 + \cdots + \frac{n(n-1)\cdots 1}{n!}h^n.$$

Theorem 6.

$$\text{(i) } \lim_{n\to\infty} |a|^n = 0 \text{ if } |a| < 1, \qquad \text{(ii) } \lim_{n\to\infty} |a|^n = \infty \text{ if } |a| > 1,$$

$$\text{(iii) } \lim_{n\to\infty} \sqrt[n]{a} = 1 \text{ for any } a > 0, \text{ (iv) } \lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Proof. Suppose that $|a| < 1$. Then $|a| = \frac{1}{1+h}$ for some

$h > 0$. Hence $0 < |a|^n = \frac{1}{(1+h)^n} < \frac{1}{1+nh}$ by lemma 1 (i). From this and Theorem 2, it is clear that $\lim_{n\to\infty} |a|^n = 0$, which proves (i). Note that the Archimedean property of real numbers is used here.

Now if $|a| > 1$, then $|a| = 1 + h$ for some $h > 0$. Again by lemma 1 (i), it follows that $|a|^n = (1 + h)^n > 1 + nh$, which implies (ii).

Note that (iii) is trivial for the case $a = 1$. First we consider the case $a > 1$. Then $\sqrt[n]{a} > 1$ and hence, for all $h_n > 0$ such that $\sqrt[n]{a} = 1 + h_n$. So by Lemma 1(i), we have $a = (1 + h_n)^n > 1 + nh_n$. This implies that $0 < h_n < \frac{a-1}{n}$. Now apply Theorem 2 to obtain $\lim_{n\to\infty} h_n = 0$ from which it follows that $\lim_{n\to\infty} \sqrt[n]{a} = 1$. If $0 < a < 1$, consider $b = \frac{1}{a} > 1$. By what we have proved $\lim_{n\to\infty} \sqrt[n]{b} = 1$, and hence $\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{b}} = 1$. This proves (iii).

To prove (iv), note that $\sqrt[n]{n} > 1$ for $n > 1$ and hence $\sqrt[n]{n} = 1 + h_n$ for some $h_n > 0$. Now using Lemma 1 (ii) we obtain $n = (1 + h_n)^n > \frac{n(n-1)}{2} h_n^2$. Thus for $n > 1$ we have $0 < h_n < \frac{\sqrt{2}}{\sqrt{n-1}}$. Now use of theorem 2 yields $\lim_{n\to\infty} h_n = 0$. Hence $\lim_{n\to\infty} \sqrt[n]{n} = 1$, which completes the proof.

Remark. Using similar arguments as above we can show that $\lim_{n\to\infty} \frac{n}{a^n} = 0$ for $a > 1$.

Following Theorem uses (iii) of Theorem 6. Of course, this Theorem is easy if we use the Second Fundamental Theorem of Calculus. The beauty of Mathematics also lies in elegant arguments. The following arguments are from *R.* Courant and Fritz John (Ref. 1).

Theorem 7. *Let* $0 < a < b \in \mathbf{R}$, *and* m *be any integer. Then*

$$\int_a^b x^m dx = \begin{cases} \frac{1}{m+1}(b^{m+1} - a^{m+1} & if \quad m \neq -1, \\ \lim\limits_{n\to\infty} n\left(\sqrt[n]{\frac{b}{a}} - 1\right) & if \quad m = -1. \end{cases}$$

Proof. Let $[a,b], 0 < a < b$, and $f(x) = x^m, m$ a non-negative integer. For $n \in \mathbf{N}$ let $q = \sqrt[n]{\frac{b}{a}}$. Then $q > 1$, and $x_k = aq^k$, where $k = 0, 1, \cdots, n$ is a partition of $[a,b]$. Now let $\Delta x_k = x_{k-1} = aq^k\left(\frac{q-1}{q}\right)$, for $k = 0, 1, \cdots, n$. Note that $\Delta x_n = b\left(\frac{q-1}{q}\right)$ is the largest among the Δx's and $\lim\limits_{n\to\infty} q = 1$ by (iii) of Theorem 6, and hence $\lim\limits_{n\to\infty} \Delta x_n = 0$. Then using the right Riemann sum we have

$$\begin{aligned}
\int_a^b x^m dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (x_k)^m aq^k \frac{q-1}{q} \\
&= \lim_{n\to\infty} a^{m+1} \frac{q-1}{q} \sum_{k=1}^{n} (q^{m+1})^k \\
&= \lim_{n\to\infty} a^{m+1} \frac{q-1}{q} q^{m+1} \left[\frac{q^{n(m+1)} - 1}{q^{m+1} - 1}\right], \text{for } m \neq -1, \\
&= \lim_{n\to\infty} (b^{m+1} - a^{m+1}) q^m \left[\frac{1}{1 + q + q^2 + \cdots + q^m}\right] \\
&= \frac{1}{m+1}(b^{m+1} - a^{m+1}).
\end{aligned}$$

Also, for $m = -1$, we have

$$\int_a^b x^m dx = \lim_{n\to\infty} \frac{n\left(\sqrt[n]{\frac{b}{a}} - 1\right)}{\sqrt[n]{\frac{b}{a}}} = \lim_{n\to\infty} n\left(\sqrt[n]{\frac{b}{a}} - 1\right).$$

This proves the Theorem for integers $m \geq 0$. Similar arguments yield the proof for m a negative integer.

Another important limit in Calculus (and in fact in the whole of Mathematics) is $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. We shall not discuss the proof of this; but at this point the beginning students of Calculus may be convinced of this by using L'Hospital's rule.

Problem Solving. Problem solving skill is very important in learning and mastering mathematical concepts and ideas. Developing problems solving skills has three important components. I tell my students that solving a mathematical problem is almost like making a trip by driving a vehicle. There are three important stages.

Trip: Knowing where you are and what your destination is.

Mathematics: Finding out what exactly is the given information, and what exactly needed to be done.

Trip: Finding out what route to take to reach the destination from the starting point.

Mathematics: Finding out how to use the given information to achieve the solution for the given problem.

Trip: Having the proper driving skills to make the trip.

Mathematics: Having the proper mathematical skills to carry out the computation.

Web Resources. The following web resources contain some tools that can be used to enhance the students' understanding. Recently, Mathematical Association of

America has started Journal of Online Mathematics and its Applications, referred to as "**JOMA**". This is a web based journal, which also contains teaching tools like Mathlets and Applets relating to Mathematics. Here is the web address for JOMA: `www.joma.org`

To browse the Mathlets in there, please go to
`www.joma.org/browse.html`

Also,

`www.joma.org/offsite.html?page=http://www.univie.ac.at/future.media/moe/galerie.html`
contains many Mathlets and Applets in the learning units of the gallery.

References

1. R.Courant and F.John, Introduction to Calculus and Analysis, Vol I, John Wiley 1965.

2. Tom M.Apostol, Calculus, Volume 1 2nd edition, John Wiley 1967.

3. `http://www.joma.org/`

4. `http://www.joma.org/offsite.html?page=http://www.univie.ac.at /future.media/moe/galerie.html`

AN INTRODUCTION TO INFINITESIMAL CALCULUS (Nonstandard Analysis) [1]

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" ... there are good reasons to believe that nonstandard analysis, in some version or other, will be the Analysis of the future. "

- *Kurt Godel* (1973)

Cognitive difficulties with Calculus:

Calculus, as we see today, has encroached upon every conceivable field of scientific application. The result is that there is a need for a Calculus course in most of the school / undergraduate classes. But, are these students comfortable with the basic definitions/concepts of Calculus?

We are all aware of the classical, and still very popular, notation for derivatives. Let f be a given function; then the derivative f' is often represented by

$$\frac{df(x)}{dx}$$

For example, the symbol

$$\frac{dx^2}{dx}$$

[1] Lecture delivered at the One day seminar on Calculus held on 10-11-2001 at I.I.T. (Madras), Chennai.

denotes the derivative of the function $f(x) = x^2$. It is clearly proclaimed that the d's are not separate entities, they should not be "cancelled" and the entire expression is *not* the quotient of the numbers "$df(x)$" and "dx". The symbol

$$\frac{dy}{dx}$$

is, no doubt, a single entity contributed by the "dynamic limit"

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \tag{1}$$

When the student studies a formula such as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{2}$$

he/she is asked to "cancel" the "du" mentally but not indicate the cancellation! David Tall [9-14] from his studies on the cognitive difficulties of learning Calculus found that interpretation of equations like (2) generates a lot of perplexity and confusion among the learners.

Schwarzenberger [6] in his book "Why Calculus cannot be made easy" points out a number of subtle concepts of the subject ill-understood by students. Even great authors of famous books may be deemed to have committed errors in the presentation of the subject, when they present it without rigorous analytical background. This is due to the nature of the subject itself whose basic notions have been viewed in different perspectives at varied points of time.

Varied approaches:

In the dynamic limit method of differentiating a real function such as

$$f(x) = x^3,$$

we assume, $h \in \mathbb{R}$, $h \neq 0$ so that

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

and as h gets closer and closer to zero, so does $3xh + h^2$ (for fixed x) and subsequently

$$\begin{aligned} f'(x) &= \lim_{x \to 0} \frac{(x+h)^3 - x^3}{h} \\ &= 3x^2 \end{aligned}$$

In order to deduce that

$$\lim_{x \to 0} \frac{(x+h)^3 - x^3}{h} = 3x^2,$$

from the classical definition, one starts with any given $\epsilon > 0$ and guarantee the existence of $\delta = \delta(\epsilon) > 0$ such that

$$0 < |h| < \delta \Rightarrow |3xh + h^2| < \epsilon.$$

In our problem, a careful computation reveals that

$$\delta = \begin{cases} \sqrt{\epsilon} & \text{for } x = 0 \\ \min\left(\frac{\epsilon}{6|x|}, \frac{\sqrt{\epsilon}}{2}\right) & \text{for } x =\neq 0. \end{cases}$$

With more complex examples, the choice of an appropriate $\delta > 0$ for a given $\epsilon > 0$ becomes inordinately laborious and difficult. In the "$\epsilon - \delta$ definition", the independent variable

(namely x in the above illustration) is static and does not move; it does snot "tend to" or "approach" a limit in any physical sense.

The good old intuitive infinitesimal method due to Sir Isaac Newton and Gottfried Leibnitz, does snot consider the limit to be the quotient but as the "value" of the quotient when h is an "infinitely small" number. This "infinitely small" quantity was denoted by dx and the "infinitely small" difference $f(x + dx) - f(x)$ by $df(x)$. Thus, if $f(x) = x^3$

$$\frac{f(x + dx) - f(x)}{dx} = 3x^2 + 3xdx + (dx)^2.$$

Now $3xdx + (dx)^2$ is an infinitesimal and consequently $3x^2+3xdx+(dx)^2$ is infinitesimally close to $3x^2$ which is the derivative. [In rough terminology, a positive infinitesimal is a number less than $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ and yet greater than zero].

Then there is the computer-drawing method in which f is graphed over a *small* interval to a highly magnified scale, and within its limits of accuracy the graph being a straight line, its gradient is the derivative.

The *numerical method* is to tabulate the values of $\frac{1}{h}[f(x+h) - f(x)]$ for a given numerical value of x so that the derivative can be easily predicted.

Infinitesimals and rigour

The approach via infinitesimals is quite intuitive and easy to assimilate, but earlier mathematicians considered it to be irrigorous , for want of a formal theoretical set up. Concepts involving infinitesimals were seen as suspect and when in the nineteenth century Calculus was placed on a firm footing

through a definition of limits by Augustin Louis Cauchy, Karl Weierstrass and others, infinitesimals were discarded and largely abandoned.

But the infinitesimals did not 'die'. Notations like $\frac{dy}{dx}$ for the derivative continued, and, sciences like Physics and Engineering even kept infinitesimals as tools. "Instants of time", "instantaneous velocities" or a curve being a series of "infinitely small line segments" or even the area bounded by a curve as an "infinite sum of infinitesimal rectangles" - all these employed clandestine use of the notion of infinitesimals.

During 1920 to 1950, there was a great explosion of Mathematical Logic. Model Theory was conceived and took a neat shape. This theory constructs and characterizes models of Axiom-systems (namely mathematical structures conforming to these axioms). For instance, Euclidean Geometry's axioms have a model in the Cartesian plane.

In the early 1960s, "the ghostly tread of infinitesimals in the corridors of Mathematics" became quite real once more, thanks to the introduction of Non-standard Analysis by logician Abraham Robinson of Yale University. Through his precise and rigorous treatment of an infinitesimal, Leibnitz's intuitive ideas were vindicated. His theory binds the Real numbers (called "standard") and additional Hyper real numbers (called "non-standard") so as to include both infinite numbers and infinitesimal numbers.

Axiomatic Systems:

Mathematical logic distinguishes between 'first order' and 'second order' axiomatic systems.

In a first-order theory, the axioms list out properties required of all objects in the system, but not all sets of objects. For example, the usual arithmetical statement (on commutativity of addition)

$$x + y = y + x \qquad \text{for all } x, y$$

is of first order.

In a second-order theory, there is no such restriction. The famous 'Archimedean axiom'

$$\text{if } \ 0 \leq x < 1/n \ \text{ for all } \ n \in N, \text{ then } x = 0 \qquad (3)$$

is of second order. This axiom rules out infinitesimals in $\mathbb{R}$.

However, if the axioms are weakened to retain only the first order properties of $\mathbb{R}$, we have other models, some that violate (3) given above. Such a system is a set of Hyper real numbers, denoted by ${}^*\mathbb{R}$. We all are aware of non-Euclidean geometry; similarly here is a non-Archimedean number system in ${}^*\mathbb{R}$.

The usual N (natural numbers), Z (integers), Q (rationals), $\mathbb{R}$ (standard reals) are all some important subsets of ${}^*\mathbb{R}$. All first order properties in $\mathbb{R}$ have a natural unique extension in ${}^*\mathbb{R}$. But second order properties like (3) are not true in ${}^*\mathbb{R}$.

Now an infinitesimal is easily defined as follows:

$x \in {}^* \mathbb{R}$. is an infinitesimal $\Leftrightarrow x \neq 0$, $0 \leq x < 1/n$ for all $n \in N$. The usual statement that "infinitesimals do not exist" is true in $\mathbb{R}$. but not in ${}^*\mathbb{R}$.

Speaking geometrically, the real line $\mathbb{R}$ is extended to a hyper real line ${}^*\mathbb{R}$, which contains infinite numbers

(infinities) and infinitesimals. Functions f defined on $\mathbb{R}$ are extended to hyper real functions ${}^*f$ defined on ${}^*\mathbb{R}$. Similarly if A is a subset of $\mathbb{R}$, it is naturally extended to a subset ${}^*A$ of ${}^*\mathbb{R}$.

Extending number systems is nothing new. For example $\mathbb{R}$. is an extension of Q just to accommodate numbers of the form $\sqrt{2}; C$ is an extension of $\mathbb{R}$. to allow -1 to have a root. In the same way, Robinson extended $\mathbb{R}$ to ${}^*\mathbb{R}$.

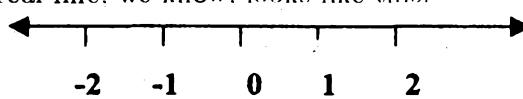
Hyper-real system

There is nothing to distinguish between $\mathbb{R}$ and ${}^*\mathbb{R}$, as far as the first order properties are concerned.

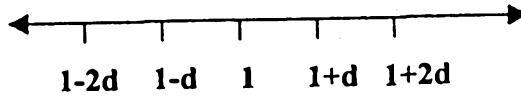
A hyper-real number is *finite* if it is smaller than some standard real; that is it lies between two standard real numbers. It is an *infinitesimal* if it is smaller than all positive standard reals. (Zero is an infinitesimal!). Anything not finite is *infinite*. Anything not in $\mathbb{R}$. is *non-standard*.

The hyper-reals ${}^*\mathbb{R}$ form an ordered field containing the reals $\mathbb{R}$ as a subfield. (Unlike the reals, the hyper-reals do not form a metric space, but by virtue of their order, they carry an order topology). Every first-order logic statement that uses basic arithmetic (like addition, multiplication, comparison etc) and quantifies only over the real numbers is also true if we presume that they quantify over hyper-real numbers.

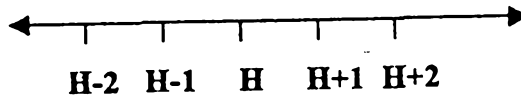
The real line, we know, looks like this:



The hyper-real line contains an infinite number H and an infinitesimal $d = 1/H$. In fact, for a given infinitesimal d we have $d = 1/H$ for some H and vice-versa. The "picture" near 1, when looked through an "*infinitesimal microscope*" would be:



distinguishing points lying at "*infinitesimal*" distance d apart. If we look at H through an "*infinitesimal telescope*" we get



illustrating "*infinities*".

Thus if x is any finite hyper real, then it has a unique standard part $st(x)$ which is infinitesimally close to x; that is $x - st(x)$ is infinitesimal. This x is uniquely expressed as

"standard real + infinitesimal".

In Robinson's Calculus, therefore, given a finite and standard x (ie., $x \in \mathbb{R}$), with a genuine (non-zero) infinitesimal ϵ,

$$f'(x) = st\left(\frac{f(x+\epsilon) - f(x)}{\epsilon}\right)$$

For example, suppose

$$f(x) = x^2$$

then

$$\begin{aligned} \frac{f(x+\epsilon)-f(x)}{\epsilon} &= \frac{(x+\epsilon)^2-x^2}{\epsilon} \\ &= 2x+\epsilon \end{aligned}$$

and thus

$$\begin{aligned} f(x) &= st(2x+\epsilon) \\ &= 2x. \end{aligned}$$

The historical process of computing with infinitesimal quantities and then ignoring them has now a strictly logical formulation.

Elegance of Non-standard Analysis:

Compare the basic concept of 'continuity' dealt with by the $\epsilon-\delta$ technique and that of Robinson.

$\epsilon-\delta$ Method

$\forall x \in D, \forall \epsilon > 0, \exists \delta(x,\epsilon) > 0$ s.t. $\forall y \in D$

$$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$$

Infinitesimal method

$$\left.\begin{array}{c} x, y \text{ are} \\ \text{infinitesimally} \\ \text{close} \end{array}\right\} \Rightarrow \left\{\begin{array}{c} f(x), f(y) \text{ are} \\ \text{infinitesimally} \\ \text{close} \end{array}\right.$$

The $\epsilon-\delta$ technique in the definition requires quantifiers and makes manipulation awesome. On the contrary, note how infinitesimals are simple to comprehend and use.

To test if $f(x) = x^2 + 2x - 1$ is continuous at $x = 1$:

[A function is continuous at (a real) r if for every hyper-real p infinitely close to r, $f(p)$ is infinitely close to $f(r)$.]

If $p \in {}^*\mathbb{R}$ is infinitesimally close to $1, p \neq 1$, then let $p = 1 + \epsilon$ where ϵ is an infinitesimal. Therefore,

$$\begin{aligned} f(p) - f(1) &= [(1+\epsilon)^2 + 2(1+\epsilon) - 1] - (1 + 2 - 1) \\ &= 4\epsilon + \epsilon^2 \end{aligned}$$

which is an infinitesimal.

Thus f is continuous at $x = 1$.

Uniform continuity can similarly be dealt with. A real function f is uniformly continuous on a subset A of $\mathbb{R}$, iff ${}^*f(x)$ is infinitesimally close to ${}^*f(y)$, for all x, y in ${}^*A$ (corresponding to A in $\mathbb{R}$, we have ${}^*A$ in ${}^*\mathbb{R}$).

Pedagogic advantages

Non-standard analysis does not lead to any conclusions about $\mathbb{R}$, which differ from standard analysis. The nicety is that it is simpler, natural and intuitively appealing. Proofs via non-standard analysis are strikingly shorter and more direct than $\epsilon - \delta$ proofs which demand complicated estimates. This requires a reorientation of pre-calculus courses. In fact Keisler, Stroyan and David Tall have tried to simplify things. In particular David Tall's astonishingly simple Super-real system is a modification of Robinson's Hyper-real system but is applicable only to analytic functions.

Generalisations of Robinson's non-standard analysis have been found useful in such diverse fields as Functional

Analysis, Number Theory, Probability, Dynamical Systems and Mathematical Economics.[16]. But the main obstacles to the wide-spread adoption of Robinson's procedure are

i) its appreciation needs a background of Mathematical Logic and

ii) scientific conservatism.

Acknowledgements: The author thanks Prof M.S. Rangachari, Director (Retd), Ramanujan Institute for Advanced Study in Mathematics, Chennai, who motivated the author to probe into the field of Nonstandard analysis, encouraged and guided him with much needed reference material and also Prof P.V.Subramaniam of IIT, Chennai for providing him with an opportunity to share these ideas with fellow teachers.

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Non-routine Problems in Mathematics

Lesson 5, Problem 1.

Problem: Find all integers n (positive, negative or zero) for which $n^2 + n + 41$ is a perfect square.

Solution. Let $n^2 + n + 41 = m^2$. Then

$$4m^2 = 4n^2 + 4n + 164 = (2n + 1)^2 + 163,$$
$$(2m + 2n + 1)(2m - 2n - 1) = 163.$$

Since 163 is a prime number we must have

$$2m + 2n + 1 = \pm 1, \pm 163,$$

and, correspondingly,

$$2m - 2n - 1 = \pm 163, \pm 1.$$

Subtracting, $4n + 2 = -162$ or 162. Hence $n = -41$ or 40.

PROF. J.N. KAPUR - A TRIBUTE

(Born on 7-9-1923 - Died on 4-9-2002)

Prof. J. N. KAPUR

With profound sorrow we report the passing away of Professor Jagat Narain Kapur, President of AMTI from 1977 to 1989, on 4th September 2002 at Delhi. Professor M.K. Singal wrote about him as early as in 1973 (Mathematics Teacher (India) volume 10 page 20):

"Simple, unassuming, energetic, dynamic, witty and ever smiling, a voracious reader, an eloquent speaker, and a prolific writer; an ontstanding researcher, a teacher *par excel-*

lence, an able administrator and an eminent educationist; a firm believer in the ideals of Gandhi, Vivekananda and Aurobindo; a true and devoted salesman of Mathematics; these are only some of the qualities of Prof. Jagat Narain Kapur."

These qualities remained in him throughout his life. His ability in inspiring leadership qualities attracted to him willing and enthusiastic workers and had made him an institution by himself.

He distinguished himself as a student of D.A.V. High School, Daryaganj (Delhi) and Hindu College, University of Delhi and won three gold medals for his record breaking performances in B.A. (Hons) and M.A. Examinations in Mathematics. He had the privilege of learning Mathematics under the great Prof. B.R. Seth at the Hindu College. Soon after graduation he served his alma mater for over 15 years and then joined the I.I.T., Kanpur in 1961. Here he taught for nearly ten years and established himself as a teacher of repute in all the broad divisions of Mathematics - Pure, Applied, Operations Research and Statistics. This was the period of active research in his career. In fact he was initiated into research by Prof. P.L. Bhatnagar (then President of AMTI), and obtained his Ph.D. degree in 1958 for his thesis on "Internal Ballistics of Guns and Rockets".

He had to his credit several hundreds of research papers published in national and international journals and had guided dozens of students for their Ph.D. degree. His research papers covered a variety of subjects such as ballistics, fluid mechanics, non-Newtonian fluid flows, magnetohydrodynamics, lubrication theory, heat transfer, queue-

ing theory, information theory, combinatorial mathematics, mathematical programming and Probability and Statistics.

He had also published a number of books on Mathematics and Mathematics Education. He had served in number of committees constituted by the Central Government and NCERT for syllabus framing and curriculum revision.

He had been associated with almost every Mathematics Society in India and also with all the major Societies abroad. He had been the editor of Mathematics Student published by Indian mathematical Society and had also presided over the IMS Conference as its President in early seventies. He was the President of AMTI (India) from 1977 to 1989 and was responsible for making it a truly national organization. It was only during his time as President that AMTI started several chapters all over the country.

Prof. Kapur served as the Vice-chancellor of Meerut University for a term from December 1970 onwards. His interest in Mathematics was such that even as Vice-chancellor he used to find time to teach a few courses in Mathematics at the post-graduate level.

The members of AMTI are grateful to Prof. J.N.Kapur for all the services rendered by him to this Association and place on record their profound sense of sorrow at his demise and convey to the members of his family their heartfelt condolences. May his soul rest in peace.

EDITOR

Mathematics Teacher.

Dr. M.S. Samuel, Professor of Mathematics, Mahendra Engineering College, Mallasamudram (W), Vadugampalayam (P.O), Trichengode Taluk, Namakkal Dist. Pin 637 503 has communicated the following note written by Mr. Ebie M.Sam, a student of II B.Sc. (Physics) of Bishop Moore College, Mavelikkara.

A Property of consecutive odd numbers which are prime

Ebie M.Sam

Theorem. *The difference between any two pairs of consecutive odd numbers which are prime, is always an integral multiple of* 6.

Explanation. Consider two consecutive odd numbers x and $(x+2)$ which are both prime.

Consider another such pair y and $(y+2)$. Then $(y-x)$ will always be divisible by 6.

Proof of the Theorem. Consider two consecutive odd numbers $(n-1)$ and $(n+1)$ which are both prime. (1)

Since $(n-1)$ is prime it is not divisible by 3. So, either of the next two numbers n or $(n+1)$ should be divisible by 3. But from (1) $(n+1)$ is prime. So, it is not divisible by 3. Therefore, clearly 'n' is divisible by 3. (2)

Also, since $(n-1)$ and $(n+1)$ are odd, [because (1)] n is always even. (3)

Thus, the even number 'n' between any two consecutive

odd numbers $(n-1)$ and $(n+1)$ which are prime, is always divisible by 3.

Conversely, a pair of consecutive odd numbers which are prime can occur *if and only if* the even number included between them is divisible by 3. (4)

Consider the next number which is divisible by 3 after n.

It is $(n+3)$

Equation (4) $\Rightarrow$ There is a possibility that the numbers on either side of $(n+3)$, i.e., $[(n+3)-1]$ and $[(n+3)+1]$ form a pair of consecutive odd numbers which are prime.

But from (3), as n is even, $(n+3)$ is odd.

Therefore, [(n+3)-1] and [(n+3)+1] are both even.
Therefore, they cannot form the required pair. (5)

The next number divisible by 3 is $(n+3)+3=(n+6)$. We have a possibility of finding the required pair in $[(n+6)-1]$ and $[(n+6)+1]$.

Since $(n+6)$ is even, the above pair is odd and can form the required pair if both numbers are prime. (6)

The next multiple of 3 is $(n+6)+3=(n+9)$. But $(n+9)$ is odd, so applying (5) it cannot give the required pair.

The next multiple of 3 is $(n+9)+3=(n+12)$. It can give the required pair $[(n+12)-1]$ and $[(n+12)+1]$ if it satisfies condition (6).

Proceeding like this, $(n+3),(n+9),(n+15),(n+21),\cdots$ would be naturally eliminated as they can never provide a pair of consecutive odd numbers which are prime.

The remaining multiples of 3 are n, $(n+6)$, $(n+12)$, $(n+18)$, $(n+24)$, $(n+30)$, $(n+36)$, $\cdots$ which may give the required pair.

Therefore, the possible pairs of consecutive odd numbers which are prime are

1. $[(n+6)-1], [(n+6)+1]$
2. $[(n+12)-1], [(n+12)+1]$
3. $[(n+18)-1], [(n+18)+1]$
4. $[(n+24)-1], [(n+24)+1]$
5. $[(n+30)-1], [(n+30)+1]$

and so on.

It is clearly seen that the interval between any of these pairs is 6 or a multiples of 6

or

The difference between any two pairs of consecutive odd numbers which are prime is always an integral multiple of 6.

This proves the theorem.

Numerical verification

Prime numbers upto 433 are given below. Consecutive odd numbers which are paired together.

[2 3]		101, 103	6	233	12	353	
5, 7	6	107, 109		239, 241		359	
11, 13		113		251		361	
17, 19		127	30	257		373	
23	12	131		263	30	379	
29, 31		137, 139		269, 271		383	72
37	12	149, 151	12	277	12	389	
41, 43		157		281, 283		397	
47		163	30	293		401	
53	18	167		307	30	409	
59, 61		173		311, 313		419, 421	12
67	12	179, 181	12	317		431, 433	
71, 73		191, 193		331			
79		197, 199	6	337	36		
83	30	211		347, 349			
89		223	30				
97		227, 229					

1. One Day Seminar on Calculus

The One Day Seminar on Calculus, sponsored by AMTI and Abacus Software Ltd., Chennai was held on the 10th November 2001. The Proceedings were held in the Seminar Hall of the Department of Mathematics, Indian Institute of Technology (Madras), Chennai. The purpose of the seminar was to highlight a few issues pertaining to the teaching of Calculus for the benefit of teacher-participants. Eighteen schools in and around Chennai deputed their teachers to the seminar. About thirty five persons participated in the seminar.

The seminar was inaugurated by Prof. G.Rangan, former Executive Chairman of the AMTI. Prof. P.V.Subrahmanyam, Executive Chairman, AMTI welcomed the participants. In the pre-lunch session Prof. R.Anantharaman (Clark Atlanta University, Georgia, USA) shared his thoughts on teaching Analytic Geometry and Calculus.

Mr. R.Athmaraman (Vivekananda Vidyalaya, Chrompet, Chennai), who spoke next, highlighted the advantages in introducing infinitesimals using Non-standard models of the number system for teaching Calculus. Mr. George Verghese, representing Abacus Software Ltd., Chennai, presented a software demonstration relevant to teaching Mathematics at the high school level.

After the lunch, Dr.G.P.Youvaraj (Emporia State University, Kansas, USA) spoke on the computation of some

basic limits needed in Calculus. Dr. V.Shankaram (formerly of NCERT) speaking after Dr. Youvaraj, elaborated on pedagogical problems in the teaching of Calculus in general and mean-value theorem in particular. The seminar concluded with a talk by Prof. P.V.Subrahmanyam (I.I.T. (Madras), Chennai) focussing on lesser known proofs of a basic limit in Calculus.

PROF. P.V. SUBRAHMANYAM
Co-ordinator

2. Workshop on Mathematics Teaching

A two day workshop on Mathematics Teaching was held on 27th and 28th July 2002 at Bharathi Vidyalaya Senior Secondary School, Perumbakkam, Chennai - 601302, sponsored6jointly by AMTI, Prof. Totadri Iyengar Memorial Educational Trust and Bharathi Vidyala Senior Secondary School. The programme was designed to equip teachers to prepare their students for major entrance tests to professional courses such as JEE for I.I.T., TNPCEE, etc. and to mathematical challenges such as Mathematics Olympiads.

During the inaugural ceremony Sri Krishna Kumar, Trustee of Sri Totadri Iyengar Educational Trust welcomed the participants and the invitees and Sri T.S. Kannan, Trustee of the same Trust spoke on the objectives and activities of the Trust in the spread of the appreciation of Mathematics, Sanskrit and Vedic studies among the students and teachers. Prof. P.V.Subrahmanyam,

Executive chairman of AMTI exhorted teachers to become members of AMTI and derive professional benefits through the Association. Mrs. K.Alamelu, Director and Correspondent of Bharathi Vidyalaya Senior Secondary School proposed a vote of thanks.

The topics covered in the workshop included Mathematical Logic, Differential Calculus and Integral Calculus and Differential Equations. During the interactive sessions, concepts were explained by illustrative examples, questions were posed and answers provided by group discussion. The resource persons were Dr.V.Shankaram (Retd. Reader, NCERT, Regional Institute of Education, Mysore), Prof. P.V.Subrahmanyam (Professor, I.I.T. (Madras), Chennai), and Dr. R.Bhaskaran (Reader, A.M.Jain College, Chennai).

For the valedictory function Sri Raja Krishnamoorthy, Consultant, Ministry of Human Resource Development was the Chief Guest. He spoke on "Joy of Learning and Teaching". In his inspiring speech he requested the teachers to show humanism in teaching to their students. Shri T.Thirumalai, Managing Trustee of Prof. T.Totadri Iyengar Educational Trust presented mementos to the resource persons, participants and the local co-ordinator.

DR. V.SHANKARAM
Academic Co-ordinator

MRS. K.ALAMELU
Local Co-ordinator

3. Mathematical Workshop at Aurangabad

Invited by Nath Valley School, Aurangabad a workshop on Mathematics was conducted by AMTI from 28-09-2002 to 30-09-2002. Eighteen teachers - eight from Nath Valley School, four from SB School and two each from Holy Cross English School, Little Flower High School and SBOA High School - participated in the workshop. The team of experts consisted of Shri M.Mahadevan, Shri R.Athmaraman and Ms. R.Vijayalakshmi.

The objective of the workshop was to draw the attention of teachers to innovative methods of teaching of Mathematics and to create interest in the students to learn, explore and enjoy Mathematics. Theorems in Geometry explained through paper folding, solution of solving non-routine problems and use of graphs in the class room teaching were some of the topics discussed. On the third day an interactive session with teachers and students of 5th and 6th standards of Nath Valley School was held.

AMTI thanks Shri Ranjith Dass, Principal of Nath Valley School for the nice arrangements made for the workshop.

PROF. M.MAHADEVAN, *Secretary*, AMTI

The name of Ms. R.Vijayalakshmi, Secretary - Projects, (36, Thyagaraya Gramani Street, T.Nagar, Chennai 600 017 (Ph. No. 8144928)) has been inadvertently left out in the list of office bearers printed on the inner backside of the cover of Vol.38 Issues 1 & 2, (2002) and Issues 3 & 4, (2002) of Mathematics Teacher. The omission is regretted.

Editor, MT.

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